Sec 1.3 part b: Existence and uniqueness of solutions

Example 1 Can you find solutions to the following IVP?

$$y' = \frac{1}{x} \quad , \quad y(0) = 2$$

Ans General Solution to $y' = \frac{1}{x}$ is

$$y = \int \frac{1}{x}\, dx + C = \ln|x| + C .$$

But $\ln|x| + C$ is not defined at $x=0$.

So this IVP has no solution.

Example 2 Can you find solutions to the following IVP?

$$y' = 2\sqrt{y} \quad , \quad y(0) = 0$$

Ans By magic, I found two functions:

$$y(x) = x^2$$

$$y(x) = 0$$

Verify that $y = x^2$ is a solution:

* $y' = 2x$
  LHS of ODE: $y' = 2x$
  RHS of ODE: $2\sqrt{y} = 2x$
  LHS equals RHS ✓
* $y(0) = 0$ ✓

Verify that $y=0$ is a solution:

* $y' = 0$
  LHS of ODE $= 0$
  RHS of ODE $= 0$ ✓
* $y(0) = 0$ ✓

Both $y = x^2$ and $y = 0$ are solutions to this IVP.

Motivation We'd like to use ODE to predict the future, so it might be useful to know whether our IVP has one and only one solution.

Existence and Uniqueness Theorem

* Part (1) [Existence]

hypothesis (1)

IF

$f(x,y)$ is continuous on some rectangle R in $\overbrace{\text{the } x\text{-}y \text{ plane}}^{\mathbb{R}^2}$ containing the point (a,b) in its interior,

THEN

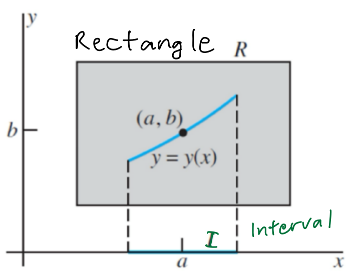
the IVP $\frac{dy}{dx} = f(x,y)$, $y(a) = b$ has at least one solution $\underbrace{\text{"near" } x = a}_{\text{on some open interval } I \text{ containing } x=a}$.

* Part (2) [Uniqueness]

hypothesis (2)

Moreover, IF $\frac{\partial f}{\partial y}$ is also continuous on the rectangle R,

THEN this solution is unique $\overbrace{\text{"near" } x = a}$. on some open interval I containing $x=a$



In other words, IF

hypothesis of the theorem

(1) $f(x,y)$ is continuous and

(2) $\frac{\partial}{\partial y} f(x,y)$ is continuous

on some rectangle R containing (a,b) in its interior,

THEN the IVP $y' = f(x,y)$, $y(a) = b$ has exactly one solution near a.

New def (partial derivatives)

To compute $\frac{\partial f}{\partial y}$, we just treat x as a constant (number) and differentiate $f(x,y)$ as if it were a function only in y.

Examples

$$\frac{\partial}{\partial y} 2\sqrt{y} = \frac{\partial}{\partial y} 2 y^{\frac{1}{2}} = 2 \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{\sqrt{y}}$$

$$\frac{\partial}{\partial y}\left(\frac{xy}{\cos x}\right) = \frac{\partial}{\partial y}\left(\frac{x}{\cos x}\right) y = \frac{x}{\cos x}$$

$$\frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial}{\partial y}(x^2) = 0$$

$$\frac{\partial}{\partial y}\left(\sqrt{x+y^2}\right) = \frac{1}{2} \frac{1}{\sqrt{x+y^2}} \cdot 2y = \frac{y}{\sqrt{x+y^2}}$$

Example 1 (again)

Can the Existence and Uniqueness Theorem be applied to guarantee that there is one unique solution to the following IVP?

$$y' = \underbrace{\frac{1}{x}}_{f(x,y)}, \quad y(\underset{a}{0}) = \underset{b}{2}$$

Answer $f(x,y) = \frac{1}{x}$ is not defined at the point $(\underset{a}{0}, \underset{b}{2})$,

so it's not continuous at $(0,2)$.

The hypotheses (1) of the theorem is not satisfied,

so we cannot apply the theorem.

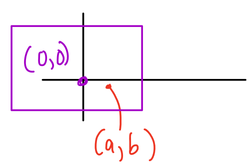
Example 2 (again)

Determine whether the Existence and Uniqueness Theorem can be applied to guarantee that there is one unique solution to the following IVP:

$$y' = \underbrace{2\sqrt{y}}_{f(x,y)}, \quad y(\underset{a}{0}) = \underset{b}{0}$$

Answer $f(x,y) = 2\sqrt{y}$

There is no rectangle containing point $(\underset{a}{0}, \underset{b}{0})$ in its interior such that $2\sqrt{y}$ is continuous.



Any rectangle containing $(0,0)$ in its interior must also contain (a,b) where b is a negative number,

but $f(a,b)$ is not defined if b is negative.

Hypothesis (1) is not satisfied, so we cannot apply the Existence and Uniqueness Thm to guarantee the existence of a solution.

WARNING: I didn't claim that no solution exists.

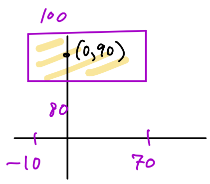
In fact, I gave you two solutions earlier

Example 3(a) Consider the IVP $y' = \sqrt{y^2 - 81}$ and $y(0) = 90$.
Determine whether the Existence and Uniqueness Theorem can be applied to guarantee that there is one unique solution to the IVP.

Ans (1) Check whether we can apply part (1) (existence) of the thm:

$f(x,y) = \sqrt{y^2 - 81}$ is continuous on a rectangle,

Say, $R := [-10, 70] \times [80, 100]$ which contains $(0, 90)$ in its interior.



Remark:
I just had to make sure my rectangle does not contain any point (x,y) where $y^2 < 81$, i.e. $y \in (-9, 9)$ because $f(x,y)$ is undefined for $y \in (-9, 9)$.

This symbol means "in"

So the IVP has a solution near $x = 0$ by part (1) of thm.

(2) Check whether we can apply part (2) (uniqueness) of the solution:

The partial derivative
$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}(y^2 - 81)^{\frac{1}{2}}$$
$$= \frac{1}{2}(y^2 - 81)^{-\frac{1}{2}} \, 2y$$
$$= \frac{y}{\sqrt{y^2 - 81}}$$

is continuous $\underbrace{\text{on the same rectangle we chose in part (1).}}_{\text{alternatively, say "at and near (0,90)"}}$

So the IVP has $\overset{\text{exactly one}}{\text{one unique}}$ solution near $x = 0$.

Q: Can we apply the Existence and Uniqueness Thm? A: Yes

Example 3(b) Can we apply the Existence & Uniqueness Thm to the IVP

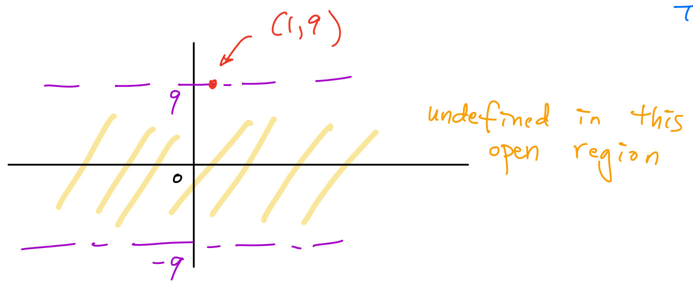
$$y' = \underbrace{\sqrt{y^2 - 81}}_{f(x,y)} \quad \text{and} \quad y(1) = 9 \quad ?$$

Ans No. We cannot apply the theorem.
(I.e. we cannot make any conclusion using the theorem.)

The hypotheses for part (1) fails

because $f(x,y) = \sqrt{y^2 - 81}$ is undefined for $y \in \underbrace{(-9, 9)}_{\text{interval from } -9 \text{ to } 9}$.

This epsilon means "in"



Any open rectangle containing the point $(1,9)$

must contain a point $(1, y)$ where $-9 < y < 9$.

So we cannot apply the theorem part (1) (existence)

Warning: We're not claiming that the IVP has no solution.

We are only claiming that the theorem cannot be applied.

(Since part (1) of thm cannot be applied,

we don't need to bother checking whether hypothesis (2) holds.

If we check hypothesis (2) anyway, we see that

$\frac{\partial}{\partial y} f(x,y)$ is not even defined at $(1,9)$.)

$\boxed{\text{Q: Can we apply the Existence and Uniqueness Thm ? A: No}}$

Example 4 (Bonus)

Determine whether the Existence and Uniqueness Theorem can be applied to guarantee that there is one unique solution to the IVP

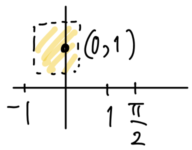
$$y' = \frac{xy}{\cos x}, \qquad y(0) = 1.$$

Short Answer: Yes

(1) Check whether we can apply part (1) (existence) of the thm.

$f(x,y) = \frac{xy}{\cos x}$ is continuous on a small rectangle containing $(0,1)$.

alternatively, say "at and near $(0,1)$"



Remark:
I just have to make sure my rectangle does not contain $\left(\frac{\pi}{2}, y\right)$ or $\left(-\frac{\pi}{2}, y\right)$ ($\frac{3.14}{2}$)

because $f(x,y)$ is undefined there due to $\cos\left(\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$.

So the IVP has a solution near $x=0$.

(2) Check whether we can apply part (2) (uniqueness) of the solution.

$\frac{\partial}{\partial y}\left(\frac{xy}{\cos x}\right) = \frac{x}{\cos x}$ is continuous at and near $(0,1)$

(on the same rectangle we chose in part (1)).

So the IVP has one unique (exactly one) solution near $x=0$.

Example 5 (Bonus)

$$y' = \sqrt{x-y}, \qquad y(2) = 2$$

Example 6 (Bonus)

$$y' = \sqrt{x-y}, \qquad y(2) = 1$$

Answer: See Recommended textbook problems #15, 16