Math 3113 Sec 1.3 part a

Slope fields & solution curves

Sec 1.2: $\frac{dy}{dx} = f(x)$ Integrate both sides

Now consider ODE $\frac{dy}{dx} = f(x,y)$

a multivariable function in two variables x, y

Examples (from Webwork):

$y' = \frac{1}{e^x} + 2y$ $\qquad \frac{dy}{dx} = y$ $\qquad \frac{dy}{dx} = 1 + y^2$

$\frac{dy}{dx} = x + y$ $\qquad \frac{dy}{dx} = \frac{xy}{\cos(x)}$ $\qquad \frac{dy}{dx} = \sqrt{y^2 - 81}$

- We cannot simply integrate both sides.
- There is no uniform method for solving this ODE.
- Today: A graphical method for constructing approximate solutions

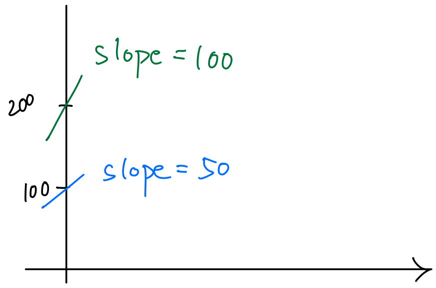
Example $\quad y' = \frac{1}{2} y$

Q: Suppose we found a solution $y(x)$ such that $y(0) = 100$. What is the slope of the tangent line to the solution $y(x)$ at $(0, 100)$?

A: The slope is $y'(0) = \frac{1}{2} 100 = 50$

Q: What if the solution we found satisfies $y(0) = 200$?

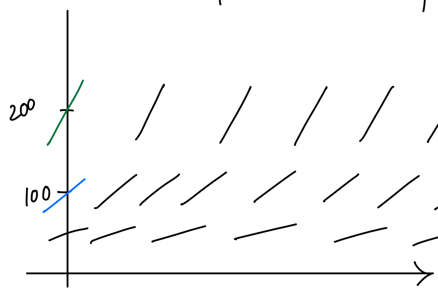
A: The slope is $y'(0) = \frac{1}{2} 200 = 100$



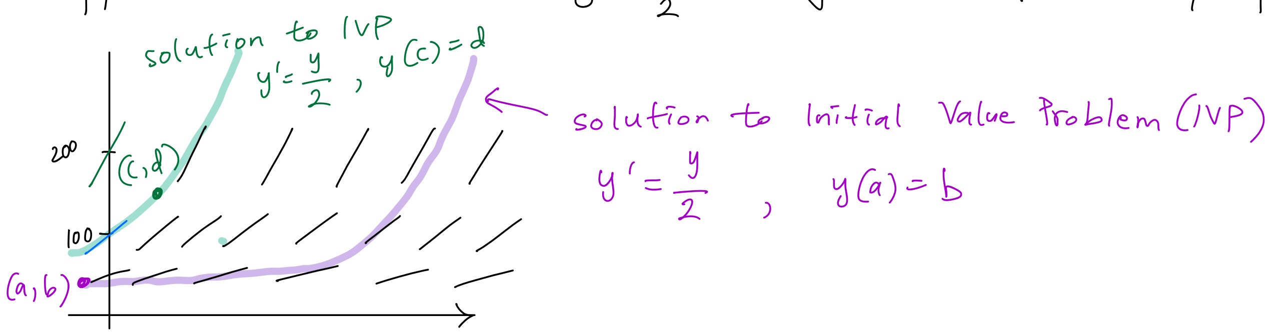
At each point (a,b), attach a small line segment w/ slope $f(a,b)=\frac{1}{2}b$.

All these small line segments form a <u>slope field</u> (or <u>direction field</u>)

- A sketch of the slope field of $\frac{dy}{dx}=\frac{y}{2}$



- Approximate solutions of $y'=\frac{y}{2}$ using sketch of the slope field:



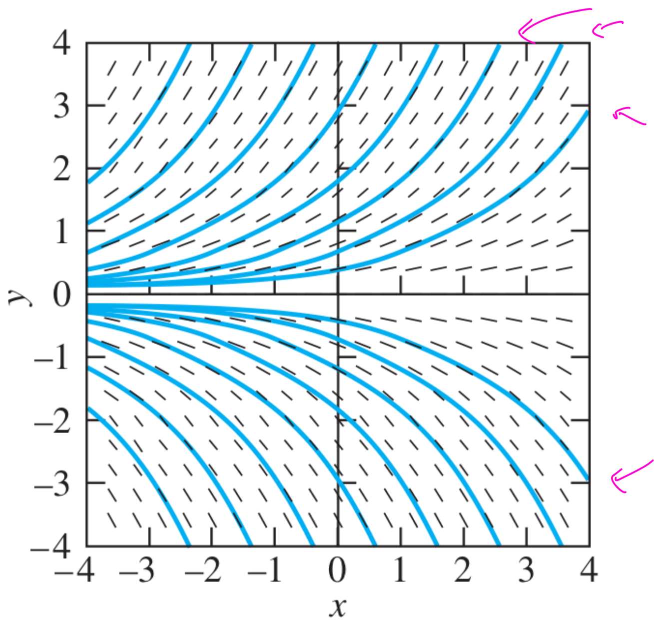


FIGURE 1.3.2(b) Slope field and solution curves for $y' = (0.5)y$.

General method for approximating graphical solutions

- In general, to construct the slope field for ODE $y' = f(x,y)$

  For each point (a,b) ...

  * compute $y'(a) = f(a,b)$
  * draw through (a,b) a small line segment w/ slope $f(a,b)$.

- Approximate solutions of $y' = f(x,y)$ using slope field.

* Choose one point (a,b) and sketch a curve by following the small line segments.
* This curve is a solution to $\frac{dy}{dx} = f(x,y)$, $y(a) = b$.

Example Consider the IVP $y' = x - y$, $y(-4) = 4$

Sketch the slope field for $y' = x - y$

Construct table of values of $f(x,y) = x - y$

$x \backslash y$	−4	−3	−2	−1	0	1	2	3	4
−4	0	−1	−2	−3	−4	−5	−6	−7	−8
−3	1	0	−1	−2	−3	−4	−5	−6	−7
−2	2	1	0	−1	−2	−3	−4	−5	−6
−1	3	2	1	0	−1	−2	−3	−4	−5
0	4	3	2	1	0	−1	−2	−3	−4
1	5	4	3	2	1	0	−1	−2	−3
2	6	5	4	3	2	1	0	−1	−2
3	7	6	5	4	3	2	1	0	−1
4	8	7	6	5	4	3	2	1	0

FIGURE 1.3.3. Values of the slope $y' = x - y$ for $-4 \leq x,\ y \leq 4$.

The more values we take, the more precise the graphical solution will be.

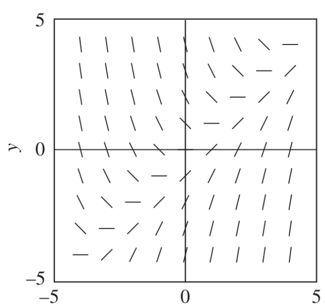


FIGURE 1.3.4. Slope field for $y' = x - y$ corresponding to the table of slopes in Fig. 1.3.3.

Solution such that $y(-4) = 4$

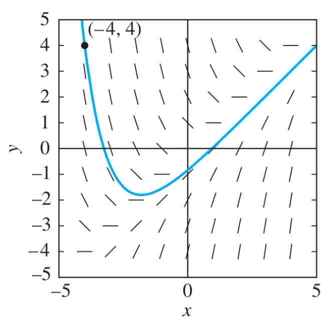


FIGURE 1.3.5. The solution curve through $(-4, 4)$.

Finer sketch

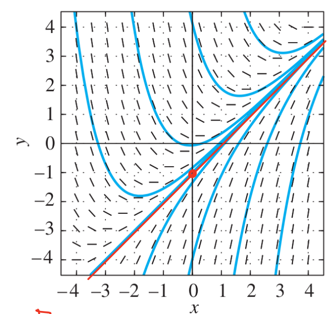


FIGURE 1.3.6. Slope field and typical solution curves for $y' = x - y$.

$y = x - 1$ is a solution such that $y(0) = -1$

Slope field plotter on GeoGebra:

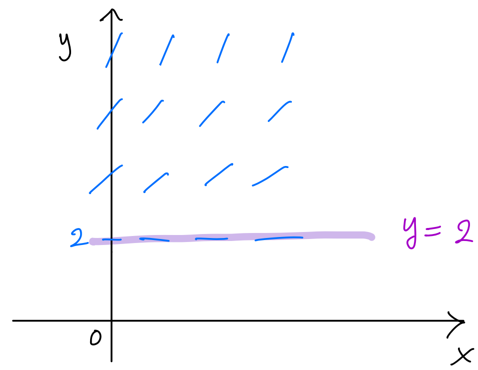
geogebra.org/m/W7dAdgqc

Try $\dfrac{dy}{dx} = xy^2$

$\dfrac{dy}{dx} = \sqrt{y-2}$

on slope field plotter

$$\frac{dy}{dx} = \sqrt{y-2} \qquad \text{slope field:}$$



$y=2$ is called an equilibrium solution

Def: Solution such that $\dfrac{dy}{dx} = 0$

(Webwork) How to associate slope fields with differential equations

Example: Match the following ODEs with their slope fields

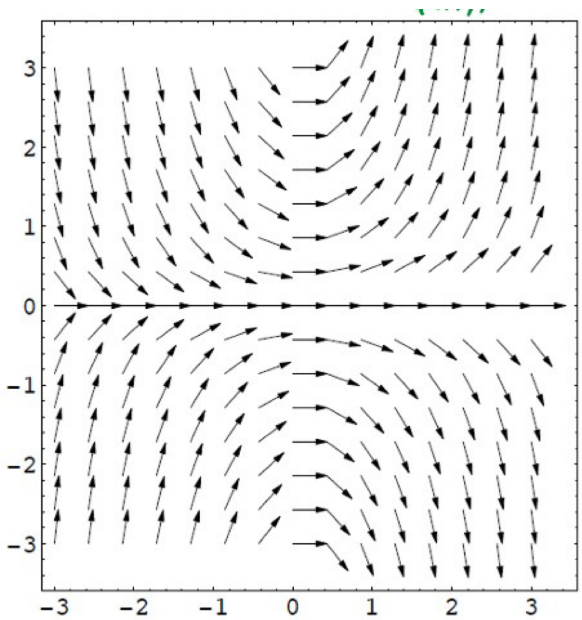


Fig I

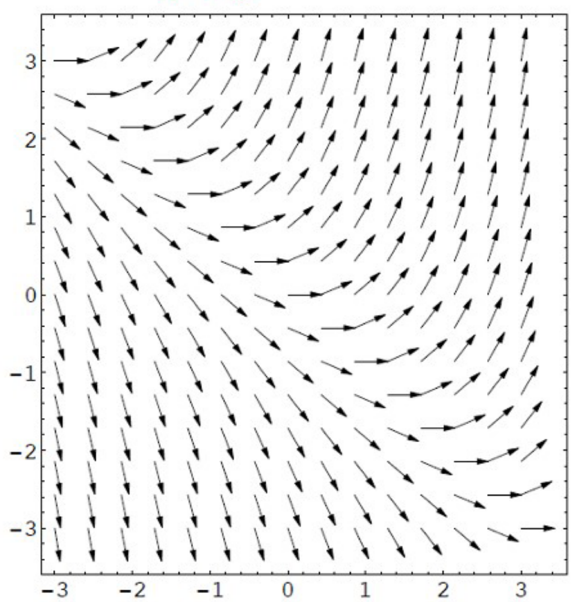


Fig II

ODE A: $\dfrac{dy}{dx} = \underbrace{x+y}_{f(x,y)}$

ODE B: $\dfrac{dy}{dx} = \underbrace{xy}_{g(x,y)}$

$\boxed{\text{Tips}}$

1) Check how derivatives behave along the x-axis by setting $y=0$ in $f(x,y)$ and $g(x,y)$.

$f(x,0) = x$

The slopes for $\frac{dy}{dx} = x+y$ are x along the x-axis.

The ODE corresponds to Fig II

$g(x,0) = 0$

The slopes for $\frac{dy}{dx} = xy$ are 0 along the x-axis

The ODE matches Fig I.

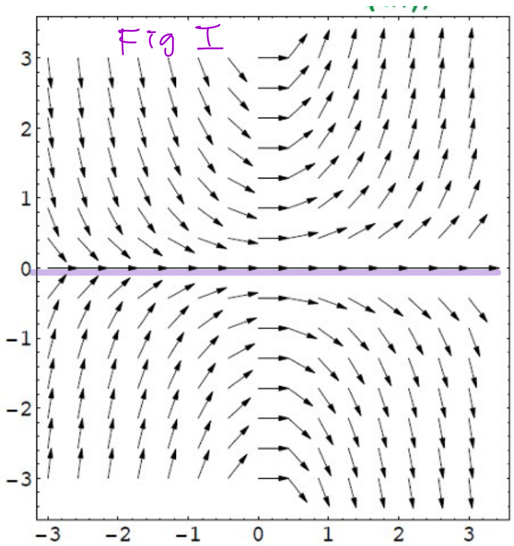


Fig I

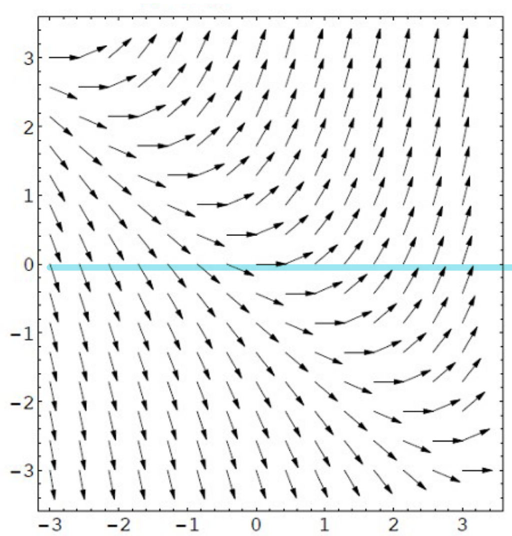


Fig II

2) Check how derivatives behave along the y-axis by setting $x=0$ in $f(x,y)$ and $g(x,y)$.

$f(0,y) = y$

The slopes for $\frac{dy}{dx} = x+y$ are y along the x-axis.

The ODE corresponds to Fig II

$g(0,y) = 0$

The slopes for $\frac{dy}{dx} = xy$ are 0 along the y-axis

The ODE matches Fig I.

3) Set $f(x,y)=0$ & $g(x,y)=0$. This gives the set of points with slope 0.

$f(x,y)=0$

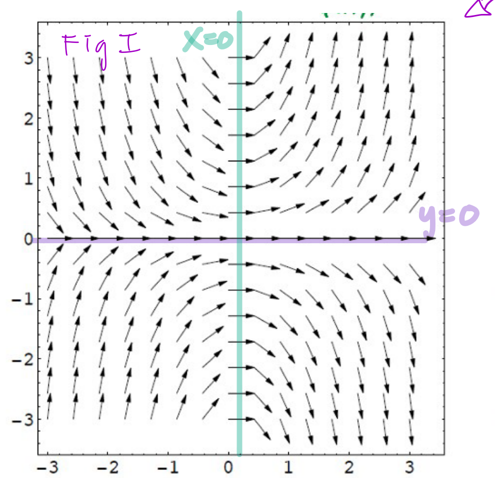
$x+y=0$

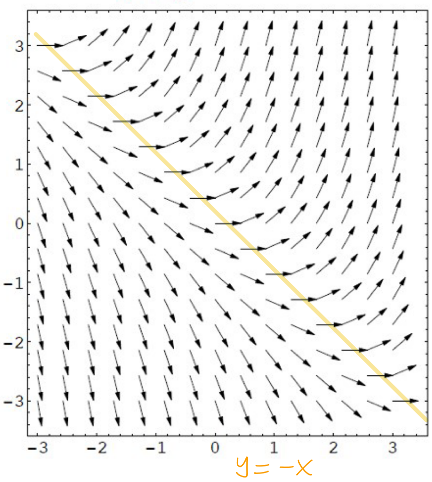
$y=-x$ (a line)

$g(x,y)=0$

$xy=0 \rightarrow x=0$, $\rightarrow y=0$

The union of two lines $x=0$ and $y=0$





4) Study the sign of $f(x,y)$ and $g(x,y)$ for different values of x and y.

$f(x,y)=x+y<0$ if $y<-x$

>0 if $y>-x$

In particular, slope at $(-1,-1)$ is $f(-1,-1)=-2<0$

$g(x,y)=xy>0$ on 1st & 3rd quadrant

<0 on 2nd & 4th quadrant.

In particular, slope at $(-1,-1)$ is $g(-1,-1)=1>0$.