Math 3113 Sec 1.2 Integral as general and particular solutions

- A function $F(x)$ is called an <u>antiderivative</u> of $f(x)$ is $(F(x))' = f(x)$.

  We often use indefinite integrals to denote antiderivatives

  e.g. $\int \cos x \, dx = \sin x + C$

- Antiderivatives of power functions:

  If $p \neq 1$, then $\int x^p \, dx = \dfrac{x^{p+1}}{p+1} + C$

  $$\int \frac{1}{x} \, dx = \ln|x| + C$$

<u>Example</u> Solve the ordinary differential equation (ODE) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$.

Equivalently, find all possible antiderivatives of $\dfrac{1}{\sqrt{x}}$:

$$\begin{aligned} y &= \int \frac{1}{\sqrt{x}} \, dx + C \\ &= \int x^{-\frac{1}{2}} \, dx + C \\ &= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= 2x^{\frac{1}{2}} + C \end{aligned}$$

$y = 2\sqrt{x} + C$ is called <u>the general solution</u> to $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$

$y = 2\sqrt{x} + 5$ is called <u>a particular solution</u> to $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$

$y = 2\sqrt{x}$ is also a particular solution to $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$.

Generally, if the ODE is of the form

$$\frac{dy}{dx} = \underbrace{f(x)}$$

$\left(\text{or } \frac{dv}{dt} = f(t) \text{ or } \frac{dy}{dt} = f(t) \text{ or } \frac{dK}{dt} = f(t) \right)$

- only in terms of x
- No y

then we can integrate both sides to obtain

the general solution

$$\boxed{y = \int f(x)\, dx + C}$$

Example 1

Find the general solution of $\frac{dy}{dx} = x e^{x^2}$.

Answer

$$y = \int x e^{x^2} dx + C$$

u-substitution : $\boxed{u = x^2}$

$du = 2x \; dx$ $\quad \boxed{\frac{1}{2} du = x \; dx}$

$$\int e^{x^2} x \, dx = \int e^u \tfrac{1}{2} du$$

u

$\frac{1}{2}$ du

$$= \frac{1}{2} e^u + C$$

$$= \boxed{\frac{1}{2} e^{x^2} + C}$$

Example 2

a) Find the general solution of $\frac{dy}{dx} = x e^x$

b) Find a solution to the Initial Value Problem $\frac{dy}{dx} = x e^x$, $y(0) = 5$

Answer

a) $$y = \int x e^x \, dx + C$$

Integration by parts: $\int u \, dv + \int v \, du = uv$ $\int u \, dv = uv - \int v \, du$

Let

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

$$\int \underset{u}{x} \underbrace{e^x \, dx}_{dv} = uv - \int v \, du$$

$$= x e^x - \int e^x \, dx$$

$$= \boxed{x e^x - e^x + C}$$

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b) $$y(t) = x e^x - e^x + C$$

$$5 = y(0) = 0 . e^0 - e^0 + C$$

$$5 = -1 + C \Rightarrow C = 6$$

$$\boxed{y(t) = x e^x - e^x + 6}$$

Example 3

Find the general solution of $\frac{dy}{dx} = \frac{1}{x^2+10x+24}$

Answer $\quad y = \int \frac{1}{x^2+10x+24}\,dx + C$

The function $\frac{1}{x^2+10x+24}$ is a rational function

$\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are both polynomials

so we can apply partial fraction method

Partial fraction: $\frac{1}{x^2+10x+24} = \frac{1}{(x+4)(x+6)} = \frac{A}{x+4} + \frac{B}{x+6}$

Solve for A, B:

Multiply both sides by $(x+4)(x+6)$: $\quad 1 = A(x+6) + B(x+4)$

Put in $(\text{a number})x + \begin{pmatrix}\text{another}\\ \text{number}\end{pmatrix}$ form: $\quad 1 = (A+B)x + (A6+B4)$

$$\left.\begin{aligned} 0 &= A+B \\ 1 &= 6A+4B \end{aligned}\right\} \Rightarrow \begin{aligned} &B = -A \qquad\qquad\qquad\qquad \Rightarrow B = -\tfrac{1}{2} \\ &1 = 6A + 4(-A) = 2A \Rightarrow \tfrac{1}{2} = A \end{aligned}$$

So $\frac{1}{(x+4)(x+6)} = \frac{1}{2}\frac{1}{x+4} - \frac{1}{2}\frac{1}{(x+6)}$

$$\int \frac{1}{x^2+10x+24}\,dx = \int \frac{1}{2}\frac{1}{x+4} - \frac{1}{2}\frac{1}{(x+6)}$$

$$= \boxed{\frac{1}{2}\ln|x+4| - \frac{1}{2}\ln|x+6| + C}$$

Example 4

You throw an arrow (of mass 3 kg) straight up from the ground with velocity 49 m/s. $\quad$ m/s = meters/second

When does the arrow return to the ground?

Answer

Let g be the gravity on earth $\quad g \approx 9.8 \ m/s^2$

Let $y(t)$ = height of arrow after t seconds

Let $v(t)$ = velocity of arrow after t seconds (up is positive)

So $v(0) = 49$ m/s

If we ignore air resistance,

then $\underbrace{\text{acceleration of arrow}}_{\frac{dv}{dt}} = -\underbrace{g}_{\text{gravity}}$

$$\frac{dv}{dt} = -9.8$$

→ This ODE is of the form $\frac{dv}{dt} = f(t)$ so we can integrate both sides to obtain the general solution.

$$v(t) = \int -9.8 \, dt + C$$

$$v(t) = -9.8t + C$$

Since $v(0) = 49$ m/s, we have

$$49 = -9.8(0) + C \implies C = 49$$

So $\boxed{v(t) = -9.8t + 49}$

$y(t)$ = height of arrow after t seconds

So $\frac{dy}{dt} = v(t) = -9.8t + 49$ This ODE is of the form $\frac{dy}{dt} = f(t)$ so we can integrate both sides to obtain the general solution.

$$y = \int -9.8t + 49 \, dt + C$$

$$= -9.8 \frac{t^2}{2} + 49t + C$$

Since $y(0) = 0$ m, we have

$$0 = -9.8(0) + 49(0) + C \Rightarrow C = 0$$

So $\boxed{y(t) = -9.8 \frac{t^2}{2} + 49t} = -4.9t^2 + 49t$

The arrow is on the ground

iff $y(t) = 0$

iff $-4.9t^2 + 49t = 0$

iff $4.9t(-t+10) = 0$

iff $t = 0$ or $t = 10$

$\boxed{\text{The arrow returns to the ground after 10 secs.}}$