Math 3113 Introduction & Sec 1.1

Question How do we interpret derivatives $\frac{dy}{dt}$?

Rate of change

Slope of the tangent line to the graph

* A <u>differential equation</u> is an equation involving derivatives of functions.

Examples of differential equations :

1) Newton's second law

If a force $F(t)$ acts on a particle & is directed along its line of motion $x(t)$, then

$$F(t) = m\,a(t) \quad \text{or} \quad F = ma$$

where m is the mass of the particle

a is the $\underbrace{\text{acceleration}}_{\text{2nd derivative of } x(t)}$ of the particle.

$$F(t) = m \frac{d^2x}{dt^2}$$

2) Initial value problem

An example of a <u>first-order</u> diff. eq because there is only 1st derivative in the equation

$$\frac{dy}{dt} = t\,e^{(t^2)}, \quad \underbrace{y(0) = 4}_{\text{initial condition}}$$

3) System of differential equations

$$\begin{cases} \dfrac{dx}{dt} = ax - pxy = x(a-py) \\ \\ \dfrac{dy}{dt} = -by + qxy = y(-b+qx) \end{cases}$$

a, b, p, q are positive constants

This is also called the Predator - prey model

x : prey

y : predator

4) ODE vs PDE

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$ = temperature of a thin uniform rod at point x, time t

is called a <u>partial diff. equation (PDE)</u>

because it involves partial derivatives

An equation with only "Calc 1" derivatives

(the function we take the derivative of

only depends on a single variable)

is called an <u>ordinary diff. equation (ODE)</u>

What is a <u>solution</u> to a differential equation?

<u>Def</u> A function $y = f(t)$ is a <u>solution</u> to an ODE if it satisfies the equation.

Examples

1) Is $y = 3t + t^2$ a solution to the ODE

another way of writing $\frac{dy}{dt}$ $\quad ty' - y = t^2$?

<u>Answer</u> $y = 3t + t^2$

$y' = 3 + 2t$

Plug in y and y' into the right-hand of the ODE:

RHS: $t\underbrace{(3+2t)}_{y'} - \underbrace{(3t+t^2)}_{y} = 3t + 2t^2 - 3t - t^2 = t^2$

LHS: t^2

Thus RHS = LHS, and the equation is satisfied by $y = 3t + t^2$

2) Find all values of k such that $y = e^{kt}$ is a solution to the ODE $y'' + 10y' + 21y = 0$

(I.e. such that $y = e^{kt}$ satisfies the ODE)

Note: The <u>order</u> of a diff eq is the order of the highest derivative that appears, so this is a 2nd-order ODE.

Answer

$$y(t) = e^{kt}$$

$$y'(t) = k e^{kt}$$

$$y''(t) = k^2 e^{kt}$$

I know I need to compute the 1st & 2nd derivatives because the ODE involves 1st & 2nd derivatives.

* $y(t)$ is a solution

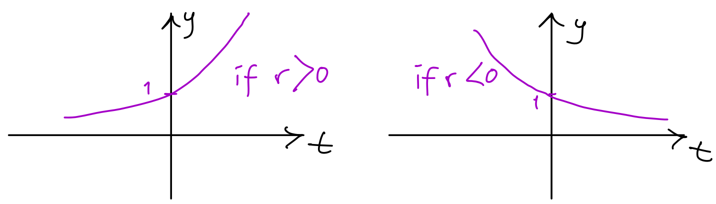
iff $$\underbrace{k^2 e^{kt}}_{y''} + 10\underbrace{k e^{kt}}_{y'} + 21\underbrace{e^{kt}}_{y'} = 0$$

Here, I plug in y, y', y'' into the ODE

iff $$e^{kt}\left(k^2 + 10k + 21\right) = 0.$$

Now I want to find all k such that $e^{kt}(k^2 + 10k + 21) = 0$

Recall $e^{kt} > 0$ for all t



Graph of $y(t) = e^{rt}$

so the equation is satisfied

iff $$k^2 + 10k + 21 = 0$$

iff $$(k+3)(k+7) = 0$$

iff $$\boxed{k = -3 \text{ or } k = -7}$$

So $y = e^{kt}$ is a solution to the ODE iff $y = e^{-3t}$ or $y = e^{-7t}$.

Warning: our ODE has more than two solutions.