Written Homework 3

There are three exercises total. Textbook and lecture notes references: Section 1.5b (mixing problems), 2.4-2.6 (Euler's and improved Euler's methods, Runge-Kutta method), Section 1.1 (first day of class)

1. Consider the cascade of two tanks A and B, with $V(A) = 200$ (gal) and $V(B) = 400$ (gal), the volumes of brine in the two tanks. Each tank also initially contains 100 lb of salt.

   Pure water flows into tank A at 10 gal/min, the mixture from tank A flows into tank B at 10 gal/min, and the mixture from tank B flows out at 10 gal/min.

   (a) Find the amount $x(t)$ of salt in tank A at time t by first setting up the ODE.

   (b) Find the amount $y(t)$ of salt in tank B at time t by first setting up the ODE.

(c) Finally, find the maximum amount of salt ever in tank B.

(d) Do your computations depend on the volumes of the tanks A and B? Why or why not?

2. Consider the IVP

$$y' = 2xy^2 \quad , \quad y(0) = 1$$

Your task is to approximate the value $y(0.5)$ of the solution at $x = 0.5$ using several methods. Starting at $(x_0, y_0) = (0, 1)$, perform the next few points necessary to get from $x = 0$ to $x = 0.5$.

(a) First, perform the classical Euler's method (Sec 2.4) using step size $h = 0.25$.

(b) Next, perform the classical Euler's method (Sec 2.4) again but using a *smaller* step size $h = 0.1$.

(c) Perform the Improved Euler's method (Sec 2.5) using the *bigger* step size $h = 0.25$.

(d) Finally, perform the Runge–Kutte method (Sec 2.6) using the same *big* step size $h = 0.25$.

(e) Since we are learning these numerical methods for the first time, I purposely picked an IVP whose exact solution we are able to compute using Chapter 1 techniques, and this exact solution is

$$y(x) = \frac{1}{1 - x^2}.$$

What is the value of the exact solution at $x = 0.5$?

(f) Rank the four approximations above (from best to worst). What observations can you make?

3. Consider the ODE $y'' + 6y' + 13y = 0$,

   (a) Is $y_1 = e^{-3x}\cos(2x)$ a solution to the ODE? Show your work, including computation for y' and y''.

   (b) Is $y_2 = e^{-3x}\sin(2x)$ a solution to the ODE? Show your work, including computation for y' and y''.

   (c) Find a particular solution of the form $y = Ae^{-3x}\cos(2x) + Be^{-3x}\sin(2x)$ that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 0$.