Math 220 **Name:** ______________________________

Spring 2017 **Take-home Exam 1**

March 23-24, 2017 **Due: 10:30-11:20am or 12:30-1pm Friday**

- My availability on Thursday:
  11:30am-1:20pm,
  3-4pm (in MCS lab Olin 326),
  5-6pm.
- My availability on Friday:
  10:30-11:20am,
  12:30-1pm.
- You may and should use the course textbook, class notes, past homework, and any study guides you have created.
- Graphing technology and calculators may and should be used to verify your calculation.
- During this take-home exam, you may *not* consult other resources (other math textbooks, other people's class notes, the internet, etc). You are not to discuss the exam with individuals other than the instructor until after all students have submitted their work.
- Please submit no more than 5 problems. If you submit more than 5, I will grade the first five.
- The grading criteria is 4 successful proofs for an A, 3 successful proofs + 1 quasi-successful proof for an A-, two (2) successful and two (2) quasi-successful proofs for a B, etc.

1. **Page 5 of the textbook** will help you with this problem.

   (a) Alice says to Bob: 'If you wear a yellow scarf, I will get a dog.' Bob never wears a yellow scarf but Alice gets a dog anyway. Mathematically speaking, was Alice's statement true or false? Explain.

   (b) *Without* changing its meaning, convert each of statement into a statement having the form

   "If P, then Q".

   You do not need to understand what each term means.

   1. A matrix is invertible provided that its determinant is nonzero.
   2. For a function to be continuous, it is sufficient that it be differentiable.
   3. For a function to be continuous, it is necessary that it be integrable.
   4. The composition of two surjective functions is always surjective.
   5. M has a zero eigenvalue whenever M is singular.
   6. Being linear is a sufficient condition for being continuous.
   7. A sequence is Cauchy only if it is bounded.
   8. C is closed and bounded only if C is compact.
   9. S is countable provided that there exists a surjection $f : \mathbb{N} \to S$. [1]
   10. S is countable only if there exists an injection $f : S \to \mathbb{N}$.
   11. Every nonempty subset S of $\mathbb{N}$ has a least member.[2]
   12. Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. [3]
   13. For each $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$. [4]

[1] Thm 2.4.10, page 86

[2] The well-ordering property of $\mathbb{N}$, see page 105.

[3] page 126

[4] This is equivalent to the Archimedean property of $\mathbb{R}$, see page 128.

2. Let's say you have a theorem. A *generalization* of it is another theorem that either assumes fewer hypotheses, or proves a stronger conclusion. For example, you could prove this theorem:

   If n is an even integer, then $n(n-1)$ is even.

   This theorem is true, but there is a generalization that is even better:

   If n is a *any* integer, then $n(n-1)$ is even.

   For ideas, you can review Sec 1.4 of the textbook.

   (a) Let $x \in \mathbb{R}$. Prove (formally) that if $0 < x < 1$, then $x^2 - 2x + 1 > 0$.

   (b) Improve the above theorem by stating and proving (formally) a generalization of it.

   Hint: You can sketch the graph $f(x) = x^2 - 2x + 1$ either by hand or using a graphing technology like `https://www.desmos.com/calculator` or a hand-held graphing calculator.

3. (3 points) Consider the following two statements. One is true, and one is false.

   A. For all subsets S and T of a universal set U, we have $U \backslash (S \backslash T) \subseteq (U \backslash S) \cup T$.

   B. For all subsets S and T of a universal set U, we have $U \backslash (S \backslash T) \subseteq (U \backslash S) \cap T$.

   1. Identify the true statement. That is, write "A is true" or "B is true". Then give a formal proof of the true statement.

      Note: You must prove directly from the definitions; Do not use any previously proven facts about sets.

      Proof. Suppose $x \in U \backslash (S \backslash T)$.

      $\square$

   2. Give a counterexample to the false statement, where $U = \mathbb{N}$. (Look up this notation on page 40).

      Hint: To brainstorm, you can draw a few Venn diagrams.

4. Please read the example on this page first. The actual problems are on the next page.

Definition 1. Let S be a set. Let $\mathcal{P}(S)$ denote the collection of all subsets of of S. This set $\mathcal{P}(S)$ is called the *power set* of S. See page 90.

Let A be the set of the six MCS 220 students. Then there are 2^6 subsets of A (including the empty set $\emptyset$ and the set A). This means $\mathcal{P}(A)$ has 64 members.

Consider an injective function

$$f : A \to \mathcal{P}(A)$$

defined by

$$\begin{aligned} f(\text{ Abbie }) &= \{ \text{ Abbie}, \text{Emma}, \text{Max } \} \\ f(\text{ Charlie }) &= A \\ f(\text{ Emma }) &= \{ \text{ Charlie}, \text{Keliyah } \} \\ f(\text{ Keliyah }) &= \{ \text{ Emma}, \text{Max } \} \\ f(\text{ Max }) &= \emptyset \\ f(\text{ Saad }) &= \{ \text{ Saad } \}. \end{aligned}$$

Alternatively, if you prefer to work with natural numbers, you can consider the set $A = \{ 1, 2, 3, 4, 5, 6 \}$ and function $f : A \to \mathcal{P}(A)$ given by

$$\begin{aligned} f(1) &= \{ 1, 3, 5 \} \\ f(2) &= A \\ f(3) &= \{ 2, 4 \} \\ f(4) &= \{ 3, 5 \} \\ f(5) &= \emptyset \\ f(6) &= \{ 6 \}. \end{aligned}$$

I would like to consider a set V which I define to be

$$V = \{ x \in A \ : \ x \notin f(x) \}.$$

This set is equal to

$$V = \{ \text{ Emma}, \text{Keliyah}, \text{Max } \}, \text{ or } V = \{ 3, 4, 5 \} \text{ if you prefer numbers.}$$

The reason Abbie, Charlie, and Saad (or $1, 2,$ and 6) are *not* members of V is because

$$\begin{aligned} \text{Abbie} &\in \{ \text{ Abbie}, \text{Emma}, \text{Max } \} = f(\text{ Abbie }), \\ \text{Charlie} &\in A = f(\text{ Charlie }), \\ \text{Saad} &\in \{ \text{ Saad } \} = f(\text{ Saad }). \end{aligned}$$

Consider another function $g : A \to \mathcal{P}(A)$ defined by

$$\begin{aligned}
g(\text{ Abbie }) &= \{ \text{ Charlie}, \text{Saad } \} \\
g(\text{ Charlie }) &= \{ \text{ Charlie}, \text{Saad } \} \\
g(\text{ Emma }) &= \emptyset \\
g(\text{ Keliyah }) &= \{ \text{ Abbie}, \text{Charlie}, \text{Keliyah } \} \\
g(\text{ Max }) &= \{ \text{ Abbie}, \text{Charlie}, \text{Max } \} \\
g(\text{ Saad }) &= \{ \text{ Charlie}, \text{Saad } \}
\end{aligned}$$

which happens to be *not* injective.

Alternatively, if you prefer to work with natural numbers, you can consider the set $A = \{ 1, 2, 3, 4, 5, 6 \}$ and function $g : A \to \mathcal{P}(A)$ given by

$$\begin{aligned}
g(1) &= \{ 2, 6 \} \\
g(2) &= \{ 2, 6 \} \\
g(3) &= \emptyset \\
g(4) &= \{ 1, 2, 4 \} \\
g(5) &= \{ 1, 2, 5 \} \\
g(6) &= \{ 2, 6 \}
\end{aligned}$$

(a) Now consider a subset W of A which I define to be

$$W = \{ x \in A : x \notin g(x) \}.$$

Write the set W below. That is, list its elements explicitly (either students' names or numbers) in the same way that I wrote the set V in the previous page.

$$W = \{\underline{\hspace{6cm}}\}.$$

(b) Define another function $h : A \to \mathcal{P}(A)$ (different from f and g) such that it is injective *and* $\{x \in A : x \notin h(x) \} = A$.

(c) Define another function $k : A \to \mathcal{P}(A)$ (different from f, g, and h) such that it is *not* injective and $\{x \in A : x \notin k(x) \} = \emptyset$.

(d) (Answer as many of the following as you can)

- How many functions are there from A to $\mathcal{P}(A)$
- How many of these are injective?
- How many are surjective?

(Hint: See the starred Sec 2.3 Exercise 5 page 78).

5. Let S be the Cartesian plane $\mathbb{R} \times \mathbb{R}$ and define the equivalence relation R on S by

$$\langle x_1, y_1 \rangle \mathrm{R} \langle x_2, y_2 \rangle \text{ iff } y_1 - 2x_1 = y_2 - 2x_2.$$

(a) Describe the equivalence class $E_{\langle 1,5 \rangle}$.

(b) Describe the collection of equivalence classes $E_{\langle a,b \rangle}$.

Hint: See the answer in the book for Exercise 13 page 60.

6. A relation R on a set S is called *anti-symmetric* if for all $a, b \in S$, $a\mathrm{R}b$ and $b\mathrm{R}a$ imply $a = b$.

   A relation R on a set S is called a *partial order* if it is reflexive, anti-symmetric, and transitive.

   CHOOSE JUST TWO OF THE FOLLOWING

   (a) Prove that the relation $\subseteq$ on the collection $\mathcal{P}(\mathbb{N})$ of subsets of $\mathbb{N}$ is a partial order.

   (b) Prove that the relation $|$ (divisibility relation) on $\mathbb{N}$ is a partial order.

   (c) Prove that the relation $\geq$ on $\mathbb{N}$ is a partial order.

   (d) If s and t are binary strings, we say s is an *initial segment* of t if t can be obtained by adding zero or more digits to the end of s. For example, 010 is an initial segment of 010111, 1 is an initial segment of 10, and 111 is an initial segment of 111. Let S be the set of all binary strings. Let R be the relation on S defined by

   $$s\mathrm{R}t \text{ iff } s \text{ is an initial segment of } t.$$

   Prove that R is a partial order.