Math 220 **Name:** ____________________

Spring 2017 **In-class Exam 3**

May, 2017

- Please write a short version of the honor code pledge and sign:

  On my honor, I pledge that ...

  ____

  ____

- **Each of your solution is legible, coherent, and not ambiguous.** Your reader should not need to reread your solution several times to find a train of thought.

  Organize your writing, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly.

Problem	Points	Score
1	2	
2	4	
3	4	
4	3	
5	3	
6	2	
7	2	
8	0	
9	0	
Total:	20	

- This test is closed-book and closed-notes. No technology (including phones or calculators) is permitted.
- There are bonus questions at the end of the test. It would be best to work first on the main test.

1. (a) (1 point) Please state the well-ordering property of $\mathbb{N}$.

   (b) (1 point) Please state the principle of mathematical induction.

2. (4 points) One of the statements is true and the other is false. For each, please state whether it is true/false and provide the proofs.

(a) Consider the following statement: For all $n \in \mathbb{N}$,

$$1^1 + 2^2 + 3^2 + \cdots + n^2 = \frac{(n+1)(7n-4)}{6}.$$

If this statement is true, prove it *by induction.* If it is false, prove it by giving a counterexample.

Proof. (Proof that the statement is true / false)

□

(b) Consider the following statement: For all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

If this statement is true, prove it *by induction.* If it is false, prove it by giving a counterexample.

Proof. (Proof that the statement is true / false)

□

3. (4 points) (a) State the completeness axiom for the real numbers $\mathbb{R}$.

(b) Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Please explain why S has a supremum. Call it $M = \sup S$.

(c) Let $k \in \mathbb{R}$ and $k > 0$. Define

$$-kS = \{-k\ x : x \in S\}.$$

Show that $-kS$ is bounded below (that is, show that it has a lower bound).

(d) Since $-kS$ is bounded below (as you have shown), it has an infimum. Please prove that $-kM = \inf(-kS)$.

4. (3 points) (a) Please state ANY TWO of the four statements of the Archimedean property (as stated in the textbook).

i. ______________________________

ii. ______________________________

(b) Let

$$B = \{0\} \cup \left\{ \frac{1}{5n} : n \in \mathbb{N} \right\}.$$

Prove that $0 = \inf B$.

Proof. (You don't need to explain that 0 is a lower bound of B)

$\square$

5. (3 points) (a) State the theorem from the textbook which says that $\mathbb{Q}$ is dense in $\mathbb{R}$.

(b) Let

$$C = \{r \in \mathbb{Q} : 5 < r\} = (5, \infty) \cap \mathbb{Q}.$$

Prove that $5 = \inf C$.

Proof. (You don't need to explain that 5 is a lower bound of C)

$\square$

6. (2 points) Let A be the set of the six MCS 220 students. Let $\mathcal{P}(A)$ denote the *power set* of A. Note that there are 2^6 subsets of A (including the empty set $\emptyset$ and the set A).

Consider a function

$$f : A \to \mathcal{P}(A)$$

defined by

$$\begin{aligned} f(\text{ Abbie }) &= \{ \text{ Abbie, Emma, Max } \} \\ f(\text{ Charlie }) &= A \\ f(\text{ Emma }) &= \{ \text{ Charlie, Keliyah } \} \\ f(\text{ Keliyah }) &= \emptyset \\ f(\text{ Max }) &= \emptyset \\ f(\text{ Saad }) &= \{ \text{ Saad } \}. \end{aligned}$$

Alternatively, if you prefer to work with natural numbers, you can consider the set $A = \{ 1, 2, 3, 4, 5, 6 \}$ and function $f : A \to \mathcal{P}(A)$ given by

$$\begin{aligned} f(1) &= \{ 1, 3, 5 \} \\ f(2) &= A \\ f(3) &= \{ 2, 4 \} \\ f(4) &= \emptyset \\ f(5) &= \emptyset \\ f(6) &= \{ 6 \}. \end{aligned}$$

Consider the set V which I define to be

$$V = \{ x \in A \ : \ x \notin f(x) \}.$$

(a) What are the elements of V? State explicitly.

(b) Is the function f defined above surjective? Give a short explanation.

(c) Is the function f defined above injective? Give a short explanation.

7. (2 points) (a) Please define an injection from $\mathbb{N}$ to its powet set $P(\mathbb{N})$.

(b) Is there a bijection between $\mathbb{N}$ and its power set $P(\mathbb{N})$? Give a short explanation.

(c) Is $P(\mathbb{N})$ countable? Give a short explanation.

8. (points (bonus)) Please write a response to each of the following.

   What three theorems did you most enjoy from the course, and why? Choose one theorem of moderate difficulty and reconstruct its proof .

(b) Reflect on your overall experience in this class by describing an interesting idea that you learned and why it was interesting.

9. (points (bonus)) Recall the following new definitions.

Definition (boundary point)**.** Let S be a subset of $\mathbb{R}$. A point $p \in \mathbb{R}$ is said to be a *boundary point of S* iff,

$$\text{for every } \epsilon > 0, \text{ we have } N(p;\epsilon) \cap S \neq \emptyset \text{ and } N(p;\epsilon) \cap \mathbb{R} \backslash S \neq \emptyset.$$

Note that a boundary point of S is not necessarily a member of S. For example, the boundary points of the interval $(2, 4]$ are 2 and 4.

Definition (accumulation point)**.** A point $p \in \mathbb{R}$ is said to be an *accumulation point (or limit point) of S* iff,

$$\text{for every } \epsilon > 0, N^*(p;\epsilon) \cap S \neq \emptyset$$

$$(\text{in other words, for all } \epsilon > 0, \text{ there exists } y \in S\backslash\{p\} \text{ such that } p - \epsilon < y < p + \epsilon)\,.$$

Note that an accumulation point of S is not necessarily a member of S. For example, the accumulation points of the interval $(2, 4]$ are *all* the numbers between 2 and 4 (inclusive).

(a) Let

$$E = \{0\} \bigcup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

What are the boundary points of E? What are the accumulation points of E?

(b) What are the boundary points of the set $\mathbb{N}$? What are the accumulation points of $\mathbb{N}$?

(c) What are the boundary points of the set $\mathbb{Q}$? What are the accumulation points of $\mathbb{Q}$?

(d) Prove just ONE of the following (equivalent) statements.

i. Suppose $x \in S$. If x is *not* a boundary point of S, then x is an accumulation point.

Proof. Suppose x is *not* a boundary point of S. By definition above, ...

$\square$

ii. Equivalently, you can prove the contrapositive statement: Suppose $x \in S$. If x is *not* an accumulation point of S, then x is a boundary point of S.

Proof. Suppose x is not an accumulation point of S. By definition above, ... $\square$

(e) Prove that the following statement is FALSE in general by providing a counterexample: For every subset S of $\mathbb{R}$, if $x \in S$ and x is an accumulation point of S, then x is *not* a boundary point of S.

Proof. Let $S =$ ...

$\square$