Binomial coefficients and q-binomial coefficients

Definition (Binomial coefficients)**.** The *binomial coefficients* are the numbers

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+2)(n-k+1)}{k!} \qquad \text{for integers } n,k \text{ with } n \geq k \geq 0.$$

Theorem 1 (Integrality of binomial coefficients)**.** *For all integers n, k with $n \geq k \geq 0$,*

$$\binom{n}{k} \in \mathbb{Z}.$$

i. Proof of Theorem 1 by combinatorics

Lemma 2 (Choosing a k-subset)**.** *For all integers k, n with $n \geq k \geq 0$, ...*

ii. Binomial Theorem

Theorem 3 (Binomial Theorem)**.** *For all $n \in \mathbb{Z} \geq 0$,*

$$(x+y)^n =$$

Proposition 4 (Counting all subsets of n)**.**

iii. Pascal's Triangle

Write all binomial coefficients so that the kth element of the nth row is $\binom{n}{k}$, starting at $n = 0$.

$$\begin{array}{lccccccccccccc}
n=0 & & & & & & & \binom{0}{0} & & & & & & \\
n=1 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
n=2 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
n=3 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
n=4 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
n=5 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
n=6 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
\qquad
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Proposition 5 (Recurrence relation)**.**

Proposition 6 (Another recurrence relation)**.**

iv. Proof of Theorem 1 by calculus

For $|x| < 1$, we have the geometric series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$
$$\frac{1}{(1-x)^m} = \left(\frac{1}{1-x}\right)^m = (1 + x + x^2 + x^3 + \dots)^m = \sum_{k=0}^{\infty} a_{m,k} x^k$$

where each $a_{m,k}$ is a positive integer because of the way power series (like polynomials) multiply.

v. Proof of Theorem 1 by group theory

Theorem (Lagrange's Theorem)**.** *If a group G of order g contains a subgroup H of order h then $g/h = |G|/|H| \in \mathbb{Z}$.*

For more details, see [BB96] or any undergraduate abstract algebra textbook.

vi. q-binomial coefficient

Definition (q-analogue of n)**.** For $n \in \mathbb{Z}_{\geq 1}$, let

$$\begin{aligned}(n)_q &= \frac{q^n - 1}{q - 1} \\ &= 1 + q + q^2 + \cdots + q^{n-1}.\end{aligned}$$

Definition (q-factorial of n)**.** Set $(0)_q! := 1$ and

$$(n)_q! := (n)_q (n-1)_q \dots (2)_q (1)_q \text{ if } n \geq 1.$$

Definition (q-binomial coefficient)**.** For integers n, k with $n \geq k \geq 0$,

$$\binom{n}{k}_q := \frac{(n)_q!}{(k)_q!\,(n-k)_q!} = \frac{(n)_q(n-1)_q \dots (n-k+2)_q(n-k+1)_q}{(k)_q!}$$

Theorem 7. *For integers n, k with $n \geq k \geq 0$, $\binom{n}{k}_q$ is a polynomial in q with coefficients that are nonnegative integers.*

Remark. All our proofs above for Theorem 1 can be adapted to give a proof that $\binom{n}{k}_q$ is a polynomial.

Today's talk was inspired by [Con]. To learn more about binomial coefficients, see [Bon17, Chapters 3-4] and [Sta11, Chapter 1]. For more details on q-binomial coefficients, see [Sta18, Chapter 6] and [Sta11, Chapter 1].

References

[BB96] J. A. Beachy and W.D Blair. *Abstract Algebra*. Waveland Press, 1996. 2nd edition, http://www.math.niu.edu/~beachy/aaol/theorems.html.

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[Sta11] R. P. Stanley. *Enumerative Combinatorics, Volume 1*. Cambridge University Press, 2011. 2nd edition, http://www-math.mit.edu/~rstan/ec/ec1.pdf.

[Sta18] R. P. Stanley. *Algebraic combinatorics*. Undergraduate Texts in Mathematics. Springer, 2018. 2nd edition.