ISOPERIMETRIC INEQUALITY

Masha Gordina

University of Connecticut

http://www.math.uconn.edu/~gordina

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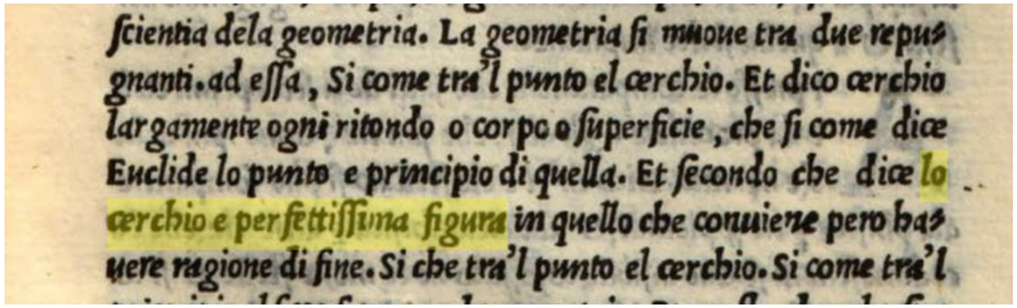
Math Club, University of Connecticut


ſcientia dela geometria. La geometria ſi muoue tra due repu-
gnanti. ad eſſa, Si come tra'l punto el cerchio. Et dico cerchio
largamente ogni ritondo o corpo o ſuperficie, che ſi come dice
Euclide lo punto e principio di quella. Et ſecondo che dice lo
cerchio e perfettiſſima figura in quello che conuiene pero ha-
uere ragione di fine. Si che tra'l punto el cerchio. Si come tra'l

Dante's *Convivio* (1531)

Gregorio Lazzarini (Venice,1655-1730) *Dido and the bull's hide*

800 BC A local ruler Iarbas (Hiarbas) agreed to sell Dido as much land as the hide of a bull could cover. She cut a bull's hide into strips and laid it out in a semi-circle around a hill (Byrsa) with the sea forming the other side. Dido then ruled Carthage as queen.

Dido Problem the *isoperimetric problem* of enclosing the maximum area within a fixed boundary

c. 300 Pappus of Alexandria, *the Collection*

1796-1863 Jakob Steiner, Switzerland. *Calculating replaces thinking while geometry stimulates it*

1870 Weierstraß gave the first rigorous proof as a corollary of *Theory of Calculus of Several Variables*

FORMULATION

- among all closed curve of length L in the plane, how large can the enclosed area be? That is, we are asking for the largest area amongst isoperimetric regions.

- Which curve (or curves) encloses the largest possible area?

- The dual question: among all regions in the plane with prescribed area A, at least how long should the perimeter be?

- Which figures realize the least perimeter?

The answers: the circle and disc.

Theorem. (**Isoperimetric Inequality**) For any region in the plane, enclosed by a piecewise C^1 boundary curve with area A and perimeter L the following holds

$$4\pi A \leqslant L^2.$$

The equality holds if and only if the curve is a circle.

The regularity of the curve can be dropped.

ERHARD SCHMIDT'S PROOF

Let $c(t) = (x(t), y(t))$ be a simple, closed, positively oriented and C^1-parameterised curve with $t \in [a, b]$.

Proposition. The area enclosed by $c(t)$ satisfies

$$A = -\int_a^b y(t)\,x'(t)\,dt = \int_a^b y'x(t)\,x(t)\,dt$$
$$= \frac{1}{2}\int_a^b \left(y'x(t)\,x(t) - y(t)\,x'(t)\right)dt$$

Proof: the first equality is Green's Theorem (Stokes' Theorem in the plane). The second equality follows from the Fundamental Theorem of Calculus. The third one follows from the first two.

Proposition. For any $(x, y, z), (a, b, c) \in \mathbb{R}^3$, then

$$(xb - za)^2 \leqslant \left(x^2 + z^2\right)\left(a^2 + b^2\right).$$

This inequality is an equality $\iff (xa + zb)^2 = 0 \iff zb = -xa$.

Proposition (Geometric/Arithmetic Mean)

For any $a, b \geqslant 0$ we have

$$\sqrt{ab} \leqslant \frac{a+b}{2}$$

with equality holding if and only if $a = b$.

Properties of parameterised curves

$c(t) = (x(t), y(t))$ is parameterized by the arclength $\Longleftrightarrow$
$|c'(t)| = 1$ for all $t \Longleftrightarrow |x'(t)|^2 + |y'(t)|^2 = 1$ for all t.

$c(t) = (x(t), y(t))$ is a circle of radius $r > 0 \Longleftrightarrow x^2 + y^2 = r^2$.

THE CALCULUS OF VARIATIONS (WEIERSTRAß)

Consider first the problem of finding a function $y(t)$ that gives the extrema of a functional such as

$$\int_a^b F(y, y', t)dt.$$

- Fundamental lemma of calculus of variations $\Longrightarrow$

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial y'} = 0,$$

the Euler–Lagrange equation.

- The isoperimetric problem: looking for the curves $(x(t), y(t))$ that maximize the area

$$\int_a^b A(x, x', y, y', t)dt = \frac{1}{2}\int_a^b \left(y'x\,(t)\,x\,(t) - y\,(t)\,x'\,(t)\right)dt$$

while the perimeter

$$\int_a^b L(x, x', y, y', t)dt = \frac{1}{2}\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$$

is fixed.

- Modification of the Euler–Lagrange equation to Euler's rule:

The problem of finding extrema for $\int_a^b A dt$ with $\int_a^b L dt$ fixed is the same as that of finding extrema for $\int_a^b (A - \lambda L)\, dt$ for some λ.

HURWITZ'S PROOF USING THE WIRTINGER INEQUALITY

The Wirtinger inequality bounds the L^2-norm of a function by the L^2-norm of its derivative. This is a version of the Poincaré Inequality. The best constant in the Poincaré Inequality is related to the spectral properties of the Laplace operator and geometry of the space.

Theorem. (**Wirtinger's inequality**) Let $f(\theta)$ be a piecewise C^1 function with period 2π, and let $\overline{f}$ denote the mean value of f

$$\overline{f} := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta.$$

Then

$$\int_0^{2\pi} \left(f(\theta) - \overline{f}\right)^2 d\theta \leqslant \int_0^{2\pi} \left(f'(\theta)\right)^2 d\theta.$$

Equality holds if and only if

$$\boldsymbol{f(\theta) = \overline{f} + a\cos\theta + b\sin\theta}$$

for some $\boldsymbol{a}$ and $\boldsymbol{b}$.

Fourier series

- Brunn-Minkowski convex analysis

- Geometric measure theory

- Polygonal geometry

- Gromov's vector analysis in *Isoperimetric inequalities in Riemannian manifolds*, 1986