Chaos: The Predictably Unpredictable

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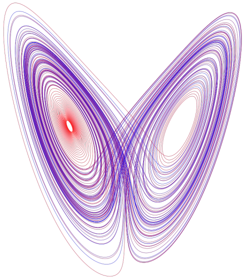
Outline

Part 1: An Introduction to **Dynamical Systems**

Part 2: What is **Chaos**?

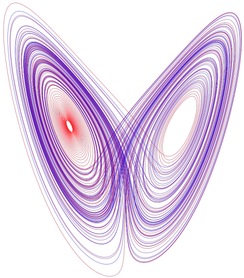
Part 3: Examples of **Chaotic Dynamics**

Dynamical Systems



Dynamical System: A system that describes a quantity that is evolving in time.

Dynamical Systems



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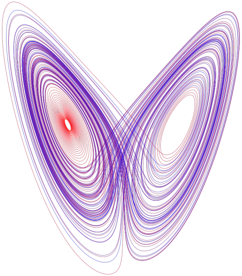

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- **Differential Equations:** Useful for quantities that evolve continuously over time.
- **Difference Equations:** Useful for quantities with generational dependence (discrete steps with finite size)

Differential Equations (DE)

- **Applications:** Physics, Engineering, Economics, etc
- Solve the **Initial Value Problem** (IVP):

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad y(1) = 1$$

- The solution is a function $y(x)$ satisfying $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ and $y(1) = 1$

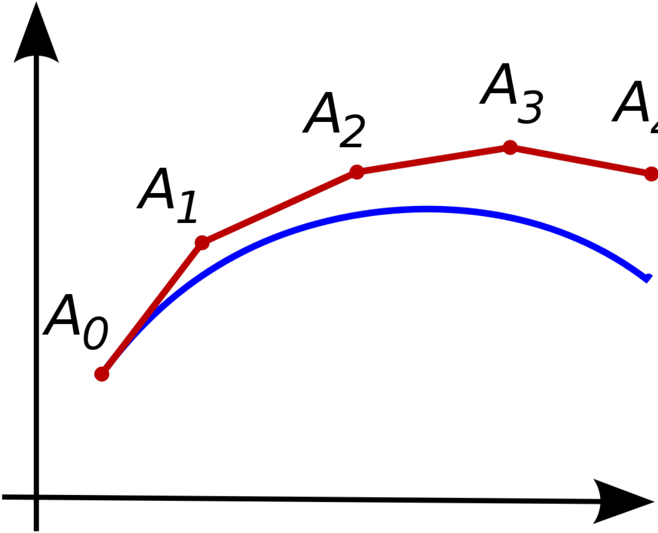
Numerical Approximations

- Most DE's cannot be solved so cleanly
- Consider the general DE:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

- Numerical Approximations with **Euler's Method**

Numerical Approximations



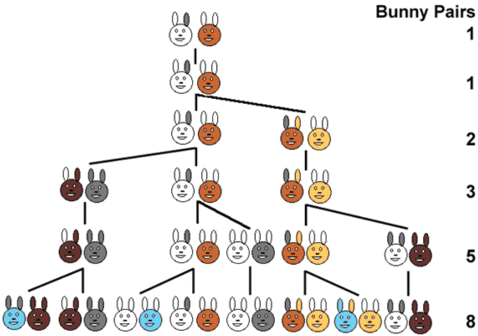
Difference Equations (ΔE)

- **Applications:** Biology, entomology, epidemiology, economics, chaos, etc.

- Can be derived from DE's
- ΔE's are **recurrence relations**
- A solution to a ΔE is a <u>sequence</u> of points

The Fibonacci Sequence



Fibonacci Sequence

$$x_{n+1} = x_n + x_{n-1}$$
$$n = 0, 1, 2, \ldots$$
$$x_0 = 1, \;\; x_{-1} = 1$$

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- **Global Dynamics:** Determine behavior of $\gamma^+(x_0)$ for any choice of x_0.

Systems of Difference Equations

- **2D System of Difference Equations:**

$$\begin{aligned} x_{n+1} &= f(x_n, y_n), \;\; n = 1, 2, \ldots \\ y_{n+1} &= g(x_n, y_n) \end{aligned}$$

- **Associated Map:** $T : \mathcal{R} \to \mathcal{R}, \; T(x, y) = (f(x, y), g(x, y))$

- **Positive Orbit:** For $(x_0, y_0) \in \mathbb{R}^2$ the positive orbit is

$$\gamma^+(x_0, y_0) = \{(x_0, y_0), T(x_0, y_0), T^2(x_0, y_0), ...\}$$

- **Global Dynamics:** Determine behavior of $\gamma^+(x_0, y_0)$ for any choice of (x_0, y_0).

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- **Stochastic:** (AKA Probabilistic) There is a probability associated with different outputs/ behaviors so the same set of parameters and initial conditions can lead to a bevy of different outputs.

Determinism: Laplace's Superman

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We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

The Art of Prediction

The ideal situation:

1. Same causes always produce the same effects.
2. Like causes always produce like effects.

The game of Pool

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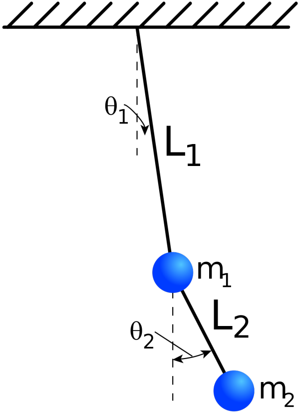
Let's see some examples!

EX 1: The Double Pendulum [Setup]



Double Pendulum

- Two masses m_1, m_2
- Two lengths L_1, L_2
- Two pivot points and corresponding angles $\theta_1(t)$ and $\theta_2(t)$
- GOAL: Track m_2 (i.e. the far tip) by tracking $\theta_1(t)$ and $\theta_2(t)$

The Double Pendulum: System of DE's

$$\theta_1'' = \frac{-g\,(2\,m_1 + m_2)\sin\theta_1 - m_2\,g\sin(\theta_1 - 2\,\theta_2) - 2\sin(\theta_1 - \theta_2)\,m_2\,(\theta_2'^2\,L_2 + \theta_1'^2\,L_1\cos(\theta_1 - \theta_2))}{L_1\,(2\,m_1 + m_2 - m_2\cos(2\,\theta_1 - 2\,\theta_2))}$$

$$\theta_2'' = \frac{2\sin(\theta_1 - \theta_2)\,(\theta_1'^2\,L_1\,(m_1 + m_2) + g(m_1 + m_2)\cos\theta_1 + \theta_2'^2\,L_2\,m_2\cos(\theta_1 - \theta_2))}{L_2\,(2\,m_1 + m_2 - m_2\cos(2\,\theta_1 - 2\,\theta_2))}$$

The Double Pendulum: Simulation

Double Pendulum Simulation

Double Pendulum, Side-by-side

Triple Pendulum Simulation

EX 2: The Gingerbread Man

2D System of ΔE's

$$x_{n+1} = 1 - y_n + |x_n|$$
$$y_{n+1} = x_n, \qquad n = 1, 2, \ldots$$

Let's check out the dynamics for some different choices of x_0, y_0.

EX 3: The Lorenz Attractor

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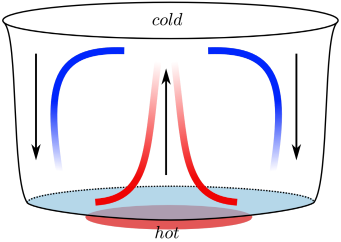

- Meteorologist Edward Lorenz, 1963 (MIT)
- Model convection in atmosphere
- 3D System of Differential Equations. Variables (x, y, z)
- Simulated with computer to track solutions (i.e how atmosphere changes)

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\begin{aligned}
\dot{x} &= \sigma(y - x) \\
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- After Lorenz ran a simulation, he went for coffee break, came back and...

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Lorenz Demo

EX 4: Fun w/ Periodic Points

- $x_{n+1} = \dfrac{1}{x_n}, \quad n = 0, 1, 2, \ldots$

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- **Period doubling** route to chaos.
- Sensitive dependence on initial conditions

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That's All Folks