GENERATING FUNCTIONS

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1. Introduction

Generating functions are certain kinds of power series that can be used to study sequences of numbers. We can apply concepts learned in calculus, such as partial fraction decompositions, to a generating function in order to help us find explicit formulas for recursively defined sequences like the Fibonacci numbers. Sequences defined in other ways have generating functions which need other ideas from real and complex analysis (radius of convergence and infinite products) in order to be studied effectively.

2. Definition and Examples

If we want to study a sequence of numbers $a_0, a_1, a_2, \dots$, we can put them all together as the coefficients of a power series.

Definition 2.1. The *generating function* of a sequence $a_0, a_1, a_2, \dots$ is the power series $f(x) = \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$.

Another power series associated to the sequence $\{a_n\}_{n\geq 0}$ is $\sum_{n\geq 0}(a_n/n!)x^n$, which is called the exponential generating function of the sequence. What we defined above, by comparison to that, is called the ordinary generating function of the sequence. Since it is the only kind we'll use, we will just use the label "generating function" as above.

Example 2.2. The constant sequence $\{1, 1, 1, \dots\}$ has generating function

$$1 + x + x^2 + x^3 + \cdots = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

This is the geometric series.

Example 2.3. For any number c, the sequence $\{1, c, c^2, c^3, \dots\}$ of powers of c has generating function

$$1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{n\geq 0} c^n x^n = \sum_{n\geq 0} (cx)^n = \frac{1}{1-cx}.$$

The previous example is the special case $c = 1$.

Example 2.4. The sequence $\{1, 2, 3, \dots\}$ of positive integers has generating function

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n\geq 0}(n+1)x^n.$$

It's not clear at first if this power series has a simple closed formula. But if we look at the terms in it, like $3x^2$, we recognize them as derivatives: $3x^2 = (x^3)'$. In fact, this power series is the derivative of

$$x + x^2 + x^3 + x^4 + \cdots = \sum_{n\geq 0} x^{n+1} = \sum_{n\geq 0} x\cdot x^n = x\sum_{n\geq 0} x^n = \frac{x}{1-x}.$$

If we differentiate $x/(1-x)$, we get $1/(1-x)^2$, so

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}.$$

The sequences appearing in these examples are ones which we think we understand pretty well (say, in terms of having simple formulas for the numbers in the sequence). The power of generating functions comes from applying them to a sequence whose terms we might not understand well. A great example of this, which we look at next, is the generating function of the Fibonacci numbers.

3. Generating Functions for Recursions

The Fibonacci numbers are defined recursively by the rules $F_1 = 1$, $F_2 = 1$, and for $n > 2$

$$F_n = F_{n-1} + F_{n-2}. \tag{3.1}$$

The first 15 Fibonacci numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610.$$

It is natural to ask if there is a formula for F_n which doesn't require us to compute the previous terms in the sequence to find it. Let's look at the generating function for the Fibonacci numbers and see what insight we get.

First we must define F_0. It is natural to choose F_0 so the recursion (3.1) holds when $n = 2$: $F_2 = F_1 + F_0$, so we define $F_0 = 0$. The generating function of the sequence $F_0, F_1, F_2, \ldots$ is

$$F(x) = \sum_{n\geq 0} F_n x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots.$$

The constant term is 0 since $F_0 = 0$. We can use the recursion (3.1) to find a simple formula for $F(x)$. Since $F_n = F_{n-1} + F_{n-2}$ when $n \geq 2$, let's separate the terms in $F(x)$ for $n \geq 2$ from the earlier terms and then apply the recursion to the coefficients:

$$\begin{aligned}
F(x) &= x + \sum_{n\geq 2} F_n x^n \\
&= x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n \\
&= x + \sum_{n\geq 2} (F_{n-1} x^n + F_{n-2} x^n) \\
&= x + \sum_{n\geq 2} F_{n-1} x^n + \sum_{n\geq 2} F_{n-2} x^n. \qquad (3.2)
\end{aligned}$$

The first series in (3.2) is

$$\sum_{n\geq 2} F_{n-1} x^n = F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots = x(F_1 x + F_2 x^2 + F_3 x^3 + \cdots) = xF(x).$$

(We used $F_0 = 0$ here.) In a similar way, the second series in (3.2) is

$$\sum_{n\geq 2} F_{n-2} x^n = x^2 F(x).$$

Substituting these into (3.2),

$$F(x) = x + xF(x) + x^2 F(x) \Longrightarrow (1 - x - x^2)F(x) = x \Longrightarrow F(x) = \frac{x}{1 - x - x^2}.$$

This is quite interesting: the generating function of the Fibonacci sequence is a simple ratio of polynomials. What makes this significant is that we can now turn around and expand

$x/(1-x-x^2)$ back into a power series in a completely different way, using partial fractions. Comparing this with the original definition of $F(x)$ as a generating function will then give us a formula for F_n.

The denominator $1-x-x^2$ can be factored as $(1-\lambda x)(1-\mu x)$, where

$$\lambda = \frac{1+\sqrt{5}}{2} \approx 1.618 \text{ and } \mu = \frac{1-\sqrt{5}}{2} \approx -.618.$$

(Here λ and μ are not the roots, but the reciprocals of the roots of $1-x-x^2$.) These numbers are related to the golden ratio $\varphi = (1+\sqrt{5})/2 \approx 1.61803713$ since $\lambda = \varphi$ and $\mu = -1/\varphi$. Writing

$$F(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\lambda x)(1-\mu x)},$$

we can decompose $F(x)$ into partial fractions:

$$\frac{x}{(1-\lambda x)(1-\mu x)} = \frac{1/\sqrt{5}}{1-\lambda x} - \frac{1/\sqrt{5}}{1-\mu x}. \tag{3.3}$$

Using the geometric series expansion $1/(1-r) = \sum_{n\geq 0} r^n$ with $r = \lambda x$ and $r = \mu x$,

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}} \sum_{n\geq 0} \lambda^n x^n - \frac{1}{\sqrt{5}} \sum_{n\geq 0} \mu^n x^n \\ &= \sum_{n\geq 0} \frac{1}{\sqrt{5}} (\lambda^n - \mu^n) x^n. \end{aligned}$$

Therefore

$$\sum_{n\geq 0} F_n x^n = \sum_{n\geq 0} \frac{1}{\sqrt{5}} (\lambda^n - \mu^n) x^n. \tag{3.4}$$

The coefficients of like powers of x on both sides of (3.4) must be equal, so

$$F_n = \frac{\lambda^n - \mu^n}{\sqrt{5}}.$$

Substituting the golden ratio in here,

$$F_n = \frac{\varphi^n - (-1/\varphi)^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \varphi^n + \frac{(-1)^{n+1}}{\sqrt{5}} \frac{1}{\varphi^n}. \tag{3.5}$$

We have obtained a formula for F_n, which was first found by de Moivre in 1720. As a check, you can verify that (3.5) recovers the known values of F_n for small n. This is tedious to work out by hand even for $n = 2$, and in fact the formula for F_n is not really of great use in computing F_n. So what good is it?

First, there is the attraction of simply having a formula, which many people find worthwhile. But more importantly, (3.5) gives us some information about the rate of growth of the Fibonacci numbers. Since $1/\varphi \approx .618$ lies between 0 and 1, the second term in the formula for F_n tends to 0 as $n \to \infty$. Therefore when n is large, $F_n \approx \frac{1}{\sqrt{5}}\varphi^n$. (In fact, since the second term in (3.5) is small enough, F_n is the nearest integer to $\frac{1}{\sqrt{5}}\varphi^n$.) So up to a scaling factor the Fibonacci numbers grow like powers of φ.

We can also use (3.5) to determine how the ratios of consecutive Fibonacci numbers, F_{n+1}/F_n, behave for large n. Since $F_n/\varphi^n \to 1/\sqrt{5}$ as $n \to \infty$ by (3.5),

$$\frac{F_{n+1}}{F_n} = \frac{F_{n+1}/\varphi^n}{F_n/\varphi^n} = \varphi\frac{F_{n+1}/\varphi^{n+1}}{F_n/\varphi^n} \to \varphi\frac{1/\sqrt{5}}{1/\sqrt{5}} = \varphi = 1.61803713\ldots.$$

For instance,

$$\frac{F_{10}}{F_9} = \frac{55}{34} \approx 1.617647 \quad \text{and} \quad \frac{F_{15}}{F_{14}} = \frac{610}{377} \approx 1.6180339.$$

Remark 3.1. When we set coefficients of x^n on both sides of (3.4) equal, we used an important property of a generating function: it has only one set of coefficients. If $f(x) = \sum_{n\geq 0} a_n x^n$, $a_n = f^{(n)}(0)/n!$ by Taylor's formula, so we can recover each coefficient of $f(x)$ from the behavior of $f(x)$ as a function. If a generating function did not determine its coefficients, we couldn't use generating functions to get formulas for the coefficients. We will return to this point in Section 5.

Our derivation of a formula for the Fibonacci numbers can be extended to any sequence which satisfies a recursion of the same basic form as the Fibonacci numbers.

Theorem 3.2. *Let $a_0, a_1, a_2, \ldots$ be a sequence satisfying a recursion*

$$a_n = Aa_{n-1} + Ba_{n-2}$$

for $n \geq 2$, where A and B are constants and $B \neq 0$. Write $1 - Ax - Bx^2 = (1-\lambda x)(1-\mu x)$, for some nonzero numbers λ and μ. If $\lambda \neq \mu$ then $a_n = c_1\lambda^n + c_2\mu^n$ for some constants c_1 and c_2.

The Fibonacci numbers are the special case $A = B = 1$ and $a_0 = 0$, $a_1 = 1$.

Proof. We look at the generating function of the sequence $\{a_n\}_{n\geq 0}$. Set

$$f(x) = \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + \sum_{n\geq 2} a_n x^n.$$

On the right side, replace a_n with $Aa_{n-1} + Ba_{n-2}$ for $n \geq 2$:

$$\begin{aligned} f(x) &= a_0 + a_1 x + \sum_{n\geq 2}(Aa_{n-1} + Ba_{n-2})x^2 \\ &= a_0 + a_1 x + Ax\sum_{n\geq 2} a_{n-1}x^{n-1} + Bx^2\sum_{n\geq 2} a_{n-2}x^{n-2}. \end{aligned}$$

Since $\sum_{n\geq 2} a_{n-1}x^{n-1} = a_1 x + a_2 x^2 + \cdots = f(x) - a_0$ and $\sum_{n\geq 2} a_{n-2}x^{n-2} = f(x)$,

$$f(x) = a_0 + a_1 x + Ax(f(x) - a_0) + Bx^2 f(x) \Longrightarrow f(x) = \frac{a_0 + (a_1 - Aa_0)x}{1 - Ax - Bx^2}.$$

(This generalizes the Fibonacci number generating function $F(x) = x/(1-x-x^2)$.) We can write $1 - Ax - Bx^2 = (1-\lambda x)(1-\mu x)$ for some λ and μ, where λ and μ are the reciprocals of the roots of $1 - Ax - Bx^2$. (The number 0 is not a root since the constant term is 0, so it makes sense to talk about reciprocals of the roots of the polynomial.) Then

$$f(x) = \frac{a_0 + (a_1 - Aa_0)x}{(1-\lambda x)(1-\mu x)}.$$

Since we are assuming $\lambda \neq \mu$, we can break this up into partial fractions:

$$f(x) = \frac{c_1}{1-\lambda x} + \frac{c_2}{1-\mu x} \tag{3.6}$$

for some constants c_1 and c_2. (The numbers c_1 and c_2 can be described explicitly in terms of λ, μ, a_0, a_1, and A, but it is slightly messy and we don't write this down here.) Expanding $1/(1-\lambda x)$ and $1/(1-\mu x)$ into geometric series in (3.6),

$$f(x) = \sum_{n\geq 0} c_1\lambda^n x^n + \sum_{n\geq 0} c_2\mu^n x^n = \sum_{n\geq 0}(c_1\lambda^n + c_2\mu^n)x^n,$$

so

$$a_n = c_1\lambda^n + c_2\mu^n.$$

□

Remark 3.3. We needed $B \neq 0$ to know $1 - Ax - Bx^2$ is quadratic (with two roots).

Generating functions can be applied to get an exact formula for sequences defined by any recursion of the form

$$a_n = A_1 a_{n-1} + A_2 a_{n-2} + \cdots + A_r a_{n-r}$$

for any r. The generating function of this sequence is a ratio of polynomials with denominator $1 - A_1x - A_2x^2 - \cdots - A_rx^r$. See [5, Chap. 4], which includes the case where the polynomial has a repeated root (not included in Theorem 3.2 when $r = 2$).

4. Generating Functions for Partitions

Consider the question: how many ways are there of making \$1 using pennies, nickels, dimes, and quarters? (We assume there is no limit on the number of available coins of each type.) With P pennies, N nickels, D dimes, and Q quarters, we have $P + 5N + 10D + 25Q$ cents. Since \$1 is 100 cents, our question is the same as asking for the number of solutions to

$$P + 5N + 10D + 25Q = 100$$

where P, N, D, and Q are nonnegative integers. This is an example of a *partition problem*: we are counting how many ways we can break up 100 into a sum of 1's, 5's, 10's and 25's.

Rather than attack just this particular problem, let's encode it in a more general problem: for any $n \geq 1$, how many ways can we form n cents out of pennies, nickels, dimes, and quarters? Let c_n be that number, so c_n is the number of solutions to

$$P + 5N + 10D + 25Q = n$$

in nonnegative integers P, N, D, and Q. (Our original problem about \$1 is the special case $n = 100$.)

For example, if $n < 5$ then we can only make n cents in one way, using n pennies, which means $c_n = 1$ for $1 \leq n \leq 4$. When $n = 5$, we could use five pennies or one nickel, so $c_5 = 2$. Similarly, $c_n = 2$ for $5 \leq n < 9$. To make 10 cents, we could use ten pennies, two nickels, one dime, or one nickel and five pennies, so $c_{10} = 4$. To find a formula for c_n for all n, we will look at the generating function of the numbers c_n:

$$C(x) = \sum_{n\geq 0} c_n x^n.$$

What is c_0? It wasn't defined. Should we set it to be 0, since we need no coins to make 0 cents? No. We set $c_0 = 1$ since there is one way of forming 0 cents out of the coins, namely using 0 coins of each type. The number of coins used is 0, but the number of ways we use no coins is 1, and that's why $c_0 = 1$.

Theorem 4.1. *The generating function of the sequence* $\{c_n\}_{n\geq 0}$ *is*

$$C(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}.$$

Proof. The key idea is to look at the effect of multiplying two power series. Let's start with

$$\frac{1}{1-x} = \sum_{P\geq 0} x^P \quad \text{and} \quad \frac{1}{1-x^5} = \sum_{N\geq 0} x^{5N}.$$

The indices of summation are deliberately written as P and N so we see more clearly below the connection to coin counting.

When we multiply the two series together,

$$\frac{1}{(1-x)(1-x^5)} = \sum_{P\geq 0} x^P \sum_{N\geq 0} x^{5N} = \sum_{P,N\geq 0} x^{P+5N}.$$

Bringing like powers of x together, we collect terms where $P + 5N$ is the same:

$$\frac{1}{(1-x)(1-x^5)} = \sum_{n\geq 0} \#\{(P,N) : P + 5N = n\} x^n.$$

The coefficients in this product count how many ways we can form n cents out of pennies and nickels. In a similar way, writing $1/(1-x^{10}) = \sum_{D\geq 0} x^{10D}$,

$$\begin{aligned} \frac{1}{(1-x)(1-x^5)(1-x^{10})} &= \sum_{P\geq 0} x^P \sum_{N\geq 0} x^{5N} \sum_{D\geq 0} x^{10D} \\ &= \sum_{P,N,D\geq 0} x^{P+5N+10D} \\ &= \sum_{n\geq 0} \#\{(P,N,D) : P + 5N + 10D = n\} x^n. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})} &= \sum_{P\geq 0} x^P \sum_{N\geq 0} x^{5N} \sum_{D\geq 0} x^{10D} \sum_{Q\geq 0} x^{25Q} \\ &= \sum_{P,N,D,Q\geq 0} x^{P+5N+10D+25Q} \\ &= \sum_{n\geq 0} \#\{(P,N,D,Q) : P + 5N + 10D + 25Q = n\} x^n \\ &= \sum_{n\geq 0} c_n x^n, \end{aligned}$$

which is the generating function $C(x)$ that we were looking for. $\square$

Using a computer algebra package, we find

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})} = 1+x+x^2+x^3+x^4+2x^5+\cdots+213x^{99}+242x^{100}+\cdots,$$

so $c_{100} = 242$. There are 242 ways of forming \$1 out of pennies, nickels, dimes, and quarters.

Getting a formula for c_n using partial fractions for $C(x)$ is complicated because the denominator in $C(x)$ has 1 as a multiple root and its degree is large. Another approach, limited to cases where the denominator is a product of terms of the form $1-x^m$, is in [3, p. 330–332].

Moving beyond coin counting, other partition problems lead to generating functions that can be products of infinitely many terms. Here are two examples.

Example 4.2. The infinite product

$$\begin{aligned}\frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\frac{1}{1-x^7}\cdots &= \sum_{n_1\geq 0} x^{n_1}\sum_{n_2\geq 0} x^{3n_2}\sum_{n_3\geq 0} x^{5n_3}\sum_{n_4\geq 0} x^{7n_4}\cdots \\ &= \sum_{n\geq 0} b_n x^n\end{aligned}$$

has for the coefficient of x^n the number of ways to write n as a sum of odd positive integers (partitions of n into odd parts): saying $n = n_1+3n_2+5n_3+7n_4+\cdots$ (with $n_k = 0$ for large k) uses n_1 1's, n_2 3's, n_3 5's, n_4 7's, and so on. For example, we can obtain 6 as a sum of 6 1's, as $1+5$, as $3+3$, and as $(1+1+1)+3$. In term of the exponents,

$$x^6 = x\cdot x^5 = x^{3\cdot 2} = x^{3\cdot 1}\cdot x^3,$$

so $b_6 = 4$.

Example 4.3. The infinite product

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots = \sum_{n\geq 0} c_n x^n$$

has for the coefficient of x^n the number of ways to write n as a sum of different positive integers (partitions of n into distinct parts). For example,

$$6 = 1+5 = 2+4 = 1+2+3,$$

so $c_6 = 4$.

There is often no formula for counting partitions of various kinds, but generating functions for partitions are nevertheless useful. Here is a striking illustration.

Theorem 4.4. *The number of partitions of n into odd parts and distinct parts are equal.*

Proof. It suffices to show the generating functions for the two kinds of partitions are equal. We computed these generating functions in the previous two examples, so we want to show

$$\frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots} \stackrel{?}{=} (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots. \tag{4.1}$$

On the left side of (4.1), insert even-degree terms $1-x^{2m}$ into the numerator and denominator, so the denominator has terms for all positive integers:

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\cdots}.$$

On the right side of this equation, each $1-x^{2m}$ in the numerator is divisible by $1-x^m$ in the denominator. Putting such terms together, we can cancel factors:

$$\begin{aligned}\frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\cdots} &= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots \\ &= (1+x)(1+x^2)(1+x^3)\cdots.\end{aligned}$$

This establishes (4.1). □

In [6, pp. 242–243], generating functions for partitions are used to prove every number has a unique binary expansion (partition of n into different powers of 2). The book [1] covers many further aspects of partitions and their generating functions.

5. Some final remarks

Our use of power series and infinite products has a serious gap: we never established domains of convergence where our manipulations of the series and products are valid. To say a power series has unique coefficients because of Taylor's formula $f^{(n)}(0)/n!$ (see Remark 3.1) requires a positive radius of convergence. At the same time, however, our generating functions were never numerically evaluated at any choice of x. The power series we used are not playing the role of functions in the strict sense but are just mathematically convenient placeholders for the coefficients.

As strange as it may seem, we don't actually need to worry about convergence of generating functions to make our previous work valid. There is a fully developed theory of power series $\sum_{n\geq 0} a_n x^n$ without convergence considerations which allows addition, multiplication, inversion, and differentiation of the series (but not numerical evaluation). This is a different way of thinking about power series than the approach taken in calculus classes, and power series from this point of view are called *formal power series*. Generating functions can usually be treated as formal power series, which makes the lack of attention to domains of convergence acceptable. For more information on this, see [4] and [5, Chap. 1].

Some generating functions do require serious attention to convergence issues and other hard analysis in order to work with them. An example is the generating function that counts representations of a positive integer as a sum of four squares. Its study uses techniques from complex analysis. See [2, pp. 24–27].

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