1 Can you compute the inverse of a square matrix?

If not, review page 6 of the pdf file lecture5b.pdf Matrix inverses: algorithm.

1.) Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ or show that it doesn't exist.

Solution: This matrix has no inverse. See row reduce computation on page 7 and 8 (Exercise 7) of the PDF file lecture5b.pdf Matrix inverses: algorithm.

2.) Find the inverse of the matrix $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ or show that it doesn't exist.

Solution: The inverse does exist. See row reduce computation on page 9 to 12 of the PDF file lecture5b.pdf Matrix inverses: algorithm.

2 Know definition of eigenvectors and how to compute them?

(If not, review lecture7b.pdf Eigenvectors.)

1.) Find all 1-eigenvectors of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$ or state that the matrix has no 1-eigenvectors.

2.) Find all 2-eigenvectors of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$ or state that the matrix has no 2-eigenvectors.

3.) Find all 3-eigenvectors of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$ or state that the matrix has no 3-eigenvectors.

Solution: All 1-eigenvalues are computed in Exercise 6 in lecture7b.pdf Eigenvectors. They can be described using one parameter.
All 2-eigenvalues and 3-eigenvectors are computed in Exercise 6 in lecture7b.pdf Eigenvectors. The 2-eigenvectors can be described using one parameter. There are no 3-eigenvectors.

3 Can you find a basis of an eigenspace of a square matrix?

If not, review lecture14a.pdf

$$\text{Let } A := \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

The number -2 is one of the eigenvalues of A. Let W be the -2-eigenspace of A.

(i) Find a basis for W. (ii) What is the dimension of W?

Solution:

$$\left[\begin{array}{ccc|c} 2-(\lambda) & 4 & 3 & 0 \\ -4 & -6-(\lambda) & -3 & 0 \\ 3 & 3 & 1-(\lambda) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2-(-2) & 4 & 3 & 0 \\ -4 & -6-(-2) & -3 & 0 \\ 3 & 3 & 1-(-2) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3/4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The only column (left of the vertical line) *without* a leading 1 is the 2nd column, so let $\boxed{x_2 = t}$. With back substitution, we have $\boxed{x_3 = 0}$ and $x_1 + x_2 + 3/4x_3 = 0$, which gives $\boxed{x_1 = -t}$.

Therefore W is the set of vectors of the form $\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$ for any t in $\mathbb{R}$. That is, W is the set of linear combination of the vector $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

(i) $\boxed{\text{A basis for } W \text{ is } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.}$

(ii) The dimension of W is $\boxed{1}$.

4 Can you use eigenbasis to speed up matrix multiplication?

If not, see Exercise 2 of lecture15a.pdf.

Let $C = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

1.) For each vector in S, determine whether it is an eigenvector of C; if so, write down the corresponding eigenvalue.

Solution: We multiply each vector of S on the right of C:

$$C\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}-4\\-8\\0\end{bmatrix} = -4\begin{bmatrix}1\\2\\0\end{bmatrix}, \qquad C\begin{bmatrix}3\\1\\0\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}, \text{ and } \qquad C\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\3\end{bmatrix} = 3\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

So all vectors of S are eigenvectors of C; the corresponding eigenvalues are $-4, 1, 3$, respectively.

2.) Is S an eigenbasis of $\mathbb{R}^3$? Explain why or why not.

Solution: Answer: Yes.

Explanation: The vectors in S are all eigenvectors of C.

We only need to verify that S is a basis of $\mathbb{R}^3$. To show this, it is sufficient to check that the concatenation matrix for S has full rank. To show this, you can perform row reduce to an REF and find that the rank of the concatenation is 3. Alternatively, you can compute that the determinant of the concatenation of S is -5 (nonzero).

3.) Compute

$$C^{999}\begin{bmatrix}10\\0\\1\end{bmatrix} = \begin{bmatrix}2 & -3 & 0\\2 & -5 & 0\\0 & 0 & 3\end{bmatrix}^{999}\begin{bmatrix}10\\0\\1\end{bmatrix}.$$

Solution: b.) First, write $\begin{bmatrix}10\\0\\1\end{bmatrix}$ as a linear combination of S:

From augmented matrix $\left[\begin{array}{ccc|c}1 & 3 & 0 & 10\\2 & 1 & 0 & 0\\0 & 0 & 1 & 1\end{array}\right]$, we get $x = -2, y = 4, z = 1$, so

$$\begin{bmatrix}10\\0\\1\end{bmatrix} = -2\begin{bmatrix}1\\2\\0\end{bmatrix} + 4\begin{bmatrix}3\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Therefore,

$$\begin{aligned}C^{999}\begin{bmatrix}10\\0\\1\end{bmatrix} &= -2C^{999}\begin{bmatrix}1\\2\\0\end{bmatrix} + 4C^{999}\begin{bmatrix}3\\1\\0\end{bmatrix} + 1C^{999}\begin{bmatrix}0\\0\\1\end{bmatrix}\\ &= -2(-4)^{999}\begin{bmatrix}1\\2\\0\end{bmatrix} + 4(1)^{999}\begin{bmatrix}3\\1\\0\end{bmatrix} + 13^{999}\begin{bmatrix}0\\0\\1\end{bmatrix}\end{aligned}$$

5 Can you diagonalize a 2×2 matrix?

If not, review lecture15b.pdf.

$$\text{Let} \quad A := \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}.$$

1.) Without actually finding an eigenbasis, show that A has an eigenbasis.

Solution: The characteristic polynomial of A is

$$\begin{aligned} \rho_A(x) &= \det(x\mathrm{Id} - A) \\ &= \det\left(\begin{bmatrix} x-3 & 5 \\ 1 & x-(-1) \end{bmatrix}\right) \\ &= (x-3)(x+1) - 5 \\ &= x^2 - 2x - 3 - 5 \\ &= x^2 - 2x - 8 \\ &= (x-4)(x+2) \end{aligned}$$

The roots of $\rho_P(x)$ are $\boxed{4}$ and $\boxed{-2}$, so A has two distinct eigenvalues. By Theorem 4 in Lecture 15b, we know that A must have an eigenbasis.

2.) Okay, now find an eigenbasis of A.

Solution: We follow Algorithm 2 from Lecture 15b: From the previous part, we know that the eigenvalues of A are 4 and -2.

We find a basis of the 4-eigenspace of A: We look for a general solution to the equation $(A - 4\mathrm{Id})\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$:

$$\left[\begin{array}{cc|c} 3-4 & 5 & 0 \\ 1 & -1-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 5 & 0 \\ 1 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -5 & 0 \\ -1 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t$, then $x = 5t$. A general solution is $t\left[\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right]$. So a basis for the 4-eigenspace of A is $\{\left[\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right]\}$

We find a basis of the -2-eigenspace of A: We look for a general solution to the equation $(A + 2\mathrm{Id})\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$:

$$\left[\begin{array}{cc|c} 3+2 & 5 & 0 \\ 1 & -1+2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 5 & 5 & 0 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 5 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t$, then $x = -t$. A general solution is $t\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$. So a basis for the -2-eigenspace of A is $\{\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]\}$.

Since there are two vectors total, the set

$$\boxed{\left\{\left[\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]\right\}}$$

is an eigenbasis of A.

3.) Use the eigenbasis of A you found in the previous question to write $A = BDB^{-1}$ **where D** $:= \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.
Then check your work by multiplying out your factorization.

Solution: Let $B := \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}$, be a concatenation of the eigenbasis.

To compute B^{-1}, use the row-reduce algorithm for general $n \times n$ matrices:

$$\left[\begin{array}{cc|cc} 5 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 5 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -6 & 1 & -5 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{-1}{6} & \frac{5}{6} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1/6 & 1/6 \\ 0 & 1 & -1/6 & 5/6 \end{array}\right]$$

So $B^{-1} = \begin{bmatrix} 1/6 & 1/6 \\ -1/6 & 5/6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$.

$$\boxed{A = BDB^{-1} = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 \\ -1/6 & 5/6 \end{bmatrix}}$$

4.) Compute A^{9999}.

Solution:

$$\text{Since} \quad A = BDB^{-1} = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 \\ -1/6 & 5/6 \end{bmatrix}$$

$$A^{100} = BD^{9999}B^{-1} = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^{9999} & 0 \\ 0 & 2^{9999} \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 \\ -1/6 & 5/6 \end{bmatrix}$$

6 Eigenvectors, eigenspace, eigenbasis, diagonalization, 3 x 3

$$\text{Let} \quad M := \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}.$$

1.) The number 2 is an eigenvalue of M, so the 2-eigenspace of M contains more than just the zero vector. Find a basis for the 2-eigenspace of M.

2.) What is the dimension of the 2-eigenspace of M?

3.) Find all 2-eigenvectors of M.

Solution: 1.) Answer key: See the last two pages of the pdf file lecture14a.pdf.
2.) A basis for the 2-eigenspace of M has two vectors, so the dimension is 2.
3.) The set of linear combinations of a basis for the 2-eigenspace of M (the basis from part (a)), *except* the zero vector.

4.) Is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ an eigenvector of M?

Solution: Do matrix multiplication $M\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, and we get $\begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$, which is equal to $3\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. So, yes, $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of M. The corresponding eigenvalue is 3.

5.) Find all 3-eigenvectors of M.

6.) The number 3 is also an eigenvalue of M, so the 3-eigenspace of M contains more than just the zero vector. Find a basis for the 3-eigenspace of M.

Solution:

$$\left[\begin{array}{ccc|c} 2-(\lambda) & 2 & 4 & 0 \\ 0 & 1-(\lambda) & -2 & 0 \\ 0 & 1 & 4-(\lambda) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2-(3) & 2 & 4 & 0 \\ 0 & 1-(3) & -2 & 0 \\ 0 & 1 & 4-(3) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 4 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 2 & 4 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The only column (left of the vertical line) *without* a leading 1 is the 3rd column, so let $\boxed{x_3 = t}$. With back substitution, we have $\boxed{x_2 = -t}$ and $x_1 + -2x_3 = 0$, which gives $\boxed{x_1 = 2t}$.

5.) So all 3-eigenvectors of M are of the form $\begin{bmatrix} 2t \\ -t \\ t \end{bmatrix}$, where t is any number except for 0.

6.) A basis for the 3-eigenspace of M is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

7.) (Do you know how to find an eigenbasis or determine that the matrix has no eigenbasis? If not, review Algorithm 2 (How to find an eigenbasis) in lecture15b.pdf.)

The matrix M has exactly two eigenvalues, 2 and 3. Find an eigenbasis for M.

Solution: Step 1 of the algorithm is already given: The matrix M has exactly two eigenvalues, 2 and 3.

Step 2: Above, we found that a basis for the A basis for the 2-eigenspace of M is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Above, we found that a basis for the 3-eigenspace of M is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Step 3: Put all vectors from a basis of each λ-eigenspace together into a set: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Since there are three vectors and M is 3×3, this set is an eigenbasis for M.

8.) Compute $M^{100} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}^{100} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: Answer key: page 10 and 11 of the pdf file lecture15a.pdf

9.) (Do you know how to diagonalize a square matrix? If not, see the theorems and examples in lecture15c.pdf Eigenbases (diagonalization).)

Is it possible to find a diagonal matrix D and a matrix B such that $M = BDB^{-1}$? Why or why not?

Solution: Since we have already computed an eigenbasis of M above, we know M has an eigenbasis. This means, yes, we can diagonalize M.

10.) If it is possible, find a diagonal matrix D and a matrix B such that $M = BDB^{-1}$.

Solution:

- For B, we can take a concatenation of any eigenbasis for M. We will take a concatenation of the eigenbasis for M we already computed above: $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

- For the entries of the diagonal matrix D, we write down the eigenvalue corresponding to each column of B: So the two leftmost of D should be 2, and the rightmost entry of D should be 3: $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
- We also need to compute the inverse of B: Use the general algorithm for computing inverse of an $n \times n$ matrix: Write down $[B \mid Id]$ then perform row reduce until the left side is an an identity matrix.

  We get $B^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$
- Theorem 5 (Diagonalizing a matrix with an eigenbasis) tells us that M should be equal to BDB^{-1}:
  $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$
- Verify your answer: Check that BDB^{-1} is in fact equal to M.

11.) Use the previous item to compute M^{1000} exactly. You can leave it as three matrices EFG (where F is a diagonal matrix).

Solution: Since $M = BDB^{-1}$, we have, for example, $M^3 = (BDB^{-1})(BDB^{-1})(BDB^{-1}) = BD^3B^{-1}$ where $D^3 = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix}$.

The answer to the question is $M^{1000} = \boxed{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{1000} & 0 & 0 \\ 0 & 2^{1000} & 0 \\ 0 & 0 & 3^{1000} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}}$.

7 Can you figure out rank and dimension without row reduce?

If not, review Exercise 5 and 6 of lecture14b.pdf.
Let

$$B := \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 1 \\ 7 & 14 & 1 \end{bmatrix}.$$

Explain your answers without doing any row reduce computation.

(a.) Find the rank of B.

Solution:
(a.) The possibilities for the rank of B are 0, 1, 2, 3.
The answer cannot be 0 because B is not the zero matrix.
The answer cannot be 1 because the third column is not a scalar multiple of the first column.
The answer cannot be 3 because the first two columns are scalar multiples of each other.
So the answer must be $\boxed{2}$.

(b.) Find the dimension of the image of B.

Solution: (b.) The dimension of the image of B is equal to the rank of B, so the answer is $\boxed{2}$.

(c.) Find the dimension of the kernel of B.

Solution: (c.) The dimension of the kernel of B is $width(B) - rank(B) = 3 - 2 = \boxed{1}$.

8 Linear independence, spanning set, basis for vectors

1.) Write down a linearly dependent spanning set of $\mathbb{R}^4$.

Solution: Strategy: Every spanning set must contain a basis, so one strategy is to write down a basis for $\mathbb{R}^4$, for example, you can write down the set which consists of all standard basis vectors for $\mathbb{R}^3$. Then add an additional vector or two to this set. The new set is still a spanning of $\mathbb{R}^4$ but the new set is not linearly independent.

2.) Write down four linearly independent vectors in $\mathbb{R}^4$ which do not form a spanning set for $\mathbb{R}^4$.

Solution: Not possible. Explanation: Since the dimension of $\mathbb{R}^4$ is 4, the 'two out of three' rule tells us that 4 linearly independent vectors in $\mathbb{R}^4$ must form a spanning set of $\mathbb{R}^4$.

3.) Write down three linearly independent vectors in $\mathbb{R}^4$ which do not form a spanning set for $\mathbb{R}^4$.

Solution: Strategy: Every spanning set must contain a basis, so one strategy is to write down a basis for $\mathbb{R}^4$, for example, you can write down the set which consists of all standard basis vectors for $\mathbb{R}^4$. Then remove one vector from this set. The new, smaller set is still linearly independent, but it's no longer a spanning set because we need at least 4 vectors to form a spanning set (since the dimension of $\mathbb{R}^4$ is 4).

9 Linear independence, spanning set, basis for polynomials

1.) Write down a linearly dependent spanning set of $\mathbb{P}_3$.

Solution: Strategy: Every spanning set must contain a basis, so one strategy is to write down a basis for $\mathbb{P}_3$. For example, you can write down the standard basis for $\mathbb{P}_3$: $\{1, x, x^2, x^3\}$.

Then add an additional polynomial or two to this set. For example, your new set may be $\{1, x, x^2, x^3, 5x^2 - 8\}$.

The new set is still a spanning of $\mathbb{P}_3$ but the new set is not linearly independent anymore; we can have at most 4 linearly independent objects in $\mathbb{P}_3$ because the dimension of $\mathbb{P}_3$ is 4.

2.) Write down four linearly independent vectors in $\mathbb{P}_3$ which do not form a spanning set for $\mathbb{P}_3$.

Solution: Not possible.

Explanation: Since the dimension of $\mathbb{P}_3$ is 4, the 'two out of three' rule tells us that 4 linearly independent vectors in $\mathbb{P}_3$ must form a spanning set of $\mathbb{P}_3$.

3.) Write down three linearly independent vectors in $\mathbb{P}_3$ which do not form a spanning set for $\mathbb{P}_3$.

Solution: Strategy: Every spanning set must contain a basis, so one strategy is to write down a basis for $\mathbb{P}_3$. For example, you can write down the standard basis for $\mathbb{P}_3$: $\{1, x, x^2, x^3\}$.

Then remove one vector from this set. The new, smaller set is still linearly independent, but it's no longer a spanning set because we need at least 4 vectors to form a spanning set (since the dimension of $\mathbb{P}_3$ is 4).

10 Linear independence, spanning set, basis for other vector spaces

1.) Write down a set of two distinct objects in the vector space $\mathcal{C}^\infty$ of all smooth functions which are linearly independent.

Solution: We showed that $\cos x$ and $\sin x$ are linearly independent, so we can write down these two functions.

Solution: We showed that e^x and e^{-x} are linearly independent, so we can write down these two functions.

2.) Does the set you wrote above forms a basis for $\mathcal{C}^\infty$? Why or why not?

Solution: No. Explanation: The vector space $\mathcal{C}^\infty$ is infinite dimensional, so any basis for $\mathcal{C}^\infty$ must have infinitely many objects. The set we wrote above only has two objects.

For example, there is no way to write the function x^2+1 as a linear combination of $\cos x$ and $\sin x$; there is no way to write the function x^2+1 as a linear combination of e^x and e^{-x}.

3.) Describe the subspace of $\mathcal{C}^\infty$ which is spanned by the set you wrote above.

Solution: By definition of spanning set, this subspace consists of all linear combinations of $\cos x$ and $\sin x$:

$$\{a\cos x + b\sin x \mid a, b \text{ are numbers }\}$$

11 Lecture 16a vector space arithmetic

1.) Write $(x-4)^3$ as a scalar multiple of $1+x+x^2$ or state that it's impossible.

Solution: It's not possible.

Computation: Exercise 1(b) on page 10 of the PDF file lecture16a.pdf

2.) Write x^2 as a linear combination of the polynomials in the set $\{1, 1+x, 1+2x+x^2\}$ or state that it's impossible.

Solution: Computation: Exercise 1(c) on page 11 of the PDF file lecture16a.pdf.

Answer: $x^2 = 1(1) - 2(1+x) + 1(1+2x+x^2)$.

12 Lecture 16b subspaces of a vector space

1.) Prove that $\{f(x) \in \mathbb{P}_2 \mid f(5) = 0\}$ is or is not a subspace.

2.) Prove that $\{f(x) \in \mathbb{P}_2 \mid f(5) = 1\}$ is or is not a subspace.

Solution: See Exercise 5(a) and (b), page 20 to 25 of lecture16b.pdf.

13 Lecture 17a linear independence and spanning set for polynomials

1.) Determine whether $\{x, x+1, x+2\}$ is linearly independent or linearly dependent.

Solution: Computation: $2(x+1) - 1(x) = x + 2$, so $2(x+1) - 1(x) - 1(x+2) = 0$.
Answer: The set is linearly dependent (in other words, not linearly independent).

2.) Determine whether $\{x, x+1, x+2\}$ is a spanning set for the vector space $\mathbb{P}_1$.

Solution: Computation: See Exercise 1(a) on page 7 of the PDF file lecture17a.pdf.
Answer: The set is a spanning set for the vector space $\mathbb{P}_1$.

3.) Determine whether $\{x - 1, x^2 - 1, x^2 - 2\}$ is linearly independent.

Solution: Computation: See Exercise 1(b) on page 8 of the PDF file lecture17a.pdf.
Alternatively, we can write $x^2 - 1 - (x^2 - x) = x^2 - 1 - x^2 + x = x - 1$, so

$$x^2 - 1 - x^2 + x - (x - 1) = 0.$$

Answer: The set is linearly dependent (in other words, not linearly independent).

4.) Determine whether $\{x^2, (x-1)^2, (x-2)^2\}$ is a basis for the vector space $\mathbb{P}_2$.

Solution: Computation: See Exercise 1(c) on page 9 of the PDF file lecture17a.pdf.
Answer: The set is a basis for the vector space $\mathbb{P}_2$.

14 Lecture 17b dimension and basis of a subspace

1.) Find the dimension of the subspace $\{f(x) \in \mathbb{P}_2 \mid f(5) = 0\}$ of $\mathbb{P}_2$.

2.) Find a basis for the subspace $\{f(x) \in \mathbb{P}_2 \mid f(5) = 0\}$ of $\mathbb{P}_2$.

Solution: See Exercise 5(a),(b), page 7,8,9 of the PDF file lecture16b.pdf.

15 Do you know how to use bases to convert elements in a general vector space into vectors?

(If not, review lecture17b.pdf starting from page 10 of the PDF file until the end.)

Compute the coefficient vector of x^2 in the basis $\{1, 1+x, 1+2x+x^2, x^3\}$ of $\mathbb{P}_3$.

Solution: Answer:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Computation: Since $\{1, 1+x, 1+2x+x^2, x^3\}$ of $\mathbb{P}_3$, we know we can write x^2 as a linear combination of the four objects in $\{1, 1+x, 1+2x+x^2, x^3\}$ in exactly one way.
Using the techniques explained in lecture16a.pdf to find that linear combination:

$$x^2 = 1(1) - 2(1+x) + 1(1+2x+x^2) + 0(x^3).$$

16 Linear transformations for vectors

Let $F : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by =

$$F(p(x)) := (x-2)p'(x)$$

1.) Determine whether $x^2 - 1$ is in the kernel of F.

Solution: Computation: We have $F(x^2 - 1) = (x-2)2x \neq 0$.
Answer: $x^2 - 1$ is NOT in the kernel of F.

2.) Determine whether 16 is in the kernel of F.

Solution: Computation: We have $F(16) = (x-2)0 = 0$.
Answer: 16 IS in the kernel of F.

3.) Determine whether $x^2 - 1$ is in the image of F.

Solution: Computation: Suppose there is a polynomial $g(x)$ in $\mathbb{P}_3$ such that $F(g(x)) = x^2 - 1$. Then $x^2 - 1 = (x-2)g'(x)$. Dividing both sides by $x - 2$, we get that $g'(x)$ is not a polynomial. So $g(x)$ cannot be a polynomial. So no such $g(x)$ exists.

Answer: $x^2 - 1$ is NOT in the image of F.

4.) Determine whether $x^2 - 4$ is in the image of F.

Solution: Computation: We wish to find a polynomial $g(x)$ in $\mathbb{P}_3$ such that $F(g(x)) = x^2 - 4$, that is, $x^2 - 4 = (x-2)g'(x)$. So we want $g(x)$ such that $g'(x) = x + 2$. For example, we can take $g(x) = \frac{1}{2}x^2 + 2x + 5$.

Answer: $x^2 - 4$ IS in the image of F because $F(\frac{1}{2}x^2 + 2x + 5) = x^2 - 4$.

5.) Find the dimension of the kernel of F. Write down a basis for the kernel of F.

Solution: The only polynomials with zero as its derivative is the constant polynomials, so the kernel of F is the set of all constant polynomials, that is, $\{r(1) \mid r \text{ is any number in } \mathbb{R}\}$.

A basis of this subspace is $\{1\}$ or $\{2\}$ or $\{16\}$, so the dimension of the kernel of F is 1.

6.) Find the dimension of the image of F. Write down a basis for the image of F.

Solution: Possible answer 1: The rank-nullity theorem says that the dimension of the domain of F is equal to $\mathsf{dim}(\mathsf{ker}(F)) + \mathsf{dim}(\mathsf{im}(F))$.

Since the domain of F is $\mathbb{P}_3$ which has dimension 4 and previously we computed $\mathsf{dim}(\mathsf{ker}(F)) = 1$, the dimension of the image of F must be $4 - 1 = 3$.

Possible answer 2: The derivative of a polynomial in $\mathbb{P}_3$ has degree 0, 1, or 2, so the $\mathsf{im}(F) = \{(x-2)h(x) \mid h(x) \in \mathbb{P}_2\}$. A basis for this subspace is $\{(x-2)1, (x-2)x, (x-2)x^2\}$, so the dimension of this subspace is 3.