1 Can you compute the inverse of a square matrix?

If not, review page 6 of the pdf file lecture5b.pdf Matrix inverses: algorithm.

1.) Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ or show that it doesn't exist.

2.) Find the inverse of the matrix $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ or show that it doesn't exist.

2 Know definition of eigenvectors and how to compute them?

(If not, review lecture7b.pdf Eigenvectors.)

1.) Find all 1-eigenvectors of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$ or state that the matrix has no 1-eigenvectors.

2.) Find all 2-eigenvectors of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$ or state that the matrix has no 2-eigenvectors.

3.) Find all 3-eigenvectors of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$ or state that the matrix has no 3-eigenvectors.

3 Can you find a basis of an eigenspace of a square matrix?

If not, review lecture14a.pdf

$$\text{Let} \quad A := \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

The number -2 is one of the eigenvalues of A. Let W be the -2-eigenspace of A.

(i) Find a basis for W. (ii) What is the dimension of W?

4 Can you use eigenbasis to speed up matrix multiplication?

If not, see Exercise 2 of lecture15a.pdf.

Let $C = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

1.) For each vector in S, determine whether it is an eigenvector of C; if so, write down the corresponding eigenvalue.

2.) Is S an eigenbasis of $\mathbb{R}^3$? Explain why or why not.

3.) Compute

$$C^{999}\begin{bmatrix}10\\0\\1\end{bmatrix} = \begin{bmatrix}2 & -3 & 0\\2 & -5 & 0\\0 & 0 & 3\end{bmatrix}^{999}\begin{bmatrix}10\\0\\1\end{bmatrix}.$$

5 Can you diagonalize a 2×2 matrix?

If not, review lecture15b.pdf.

$$\text{Let } \ A := \begin{bmatrix}3 & 5\\1 & -1\end{bmatrix}.$$

1.) Without actually finding an eigenbasis, show that A has an eigenbasis.

2.) Okay, now find an eigenbasis of A.

3.) Use the eigenbasis of A you found in the previous question to write $A = BDB^{-1}$ **where D** $:= \begin{bmatrix}4 & 0\\0 & -2\end{bmatrix}$. Then check your work by multiplying out your factorization.

4.) Compute A^{9999}.

6 Eigenvectors, eigenspace, eigenbasis, diagonalization, 3 x 3

$$\text{Let } \ M := \begin{bmatrix}2 & 2 & 4\\0 & 1 & -2\\0 & 1 & 4\end{bmatrix}.$$

1.) The number 2 is an eigenvalue of M, so the 2-eigenspace of M contains more than just the zero vector. Find a basis for the 2-eigenspace of M.

2.) What is the dimension of the 2-eigenspace of M?

3.) Find all 2-eigenvectors of M.

4.) Is $\begin{bmatrix}2\\-1\\1\end{bmatrix}$ an eigenvector of M?

5.) Find all 3-eigenvectors of M.

6.) The number 3 is also an eigenvalue of M, so the 3-eigenspace of M contains more than just the zero vector. Find a basis for the 3-eigenspace of M.

7.) (Do you know how to find an eigenbasis or determine that the matrix has no eigenbasis? If not, review Algorithm 2 (How to find an eigenbasis) in lecture15b.pdf.)

The matrix M has exactly two eigenvalues, 2 and 3. Find an eigenbasis for M.

8.) Compute $M^{100}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2 & 2 & 4\\0 & 1 & -2\\0 & 1 & 4\end{bmatrix}^{100}\begin{bmatrix}1\\1\\1\end{bmatrix}$.

9.) (Do you know how to diagonalize a square matrix? If not, see the theorems and examples in lecture15c.pdf Eigenbases (diagonalization).)

Is it possible to find a diagonal matrix D and a matrix B such that $M = BDB^{-1}$? Why or why not?

10.) If it is possible, find a diagonal matrix D and a matrix B such that $M = BDB^{-1}$.

11.) Use the previous item to compute M^{1000} exactly. You can leave it as three matrices EFG (where F is a diagonal matrix).

7 Can you figure out rank and dimension without row reduce?

If not, review Exercise 5 and 6 of lecture14b.pdf.
Let

$$B := \begin{bmatrix}1 & 2 & 1\\4 & 8 & 1\\7 & 14 & 1\end{bmatrix}.$$

Explain your answers without doing any row reduce computation.

(a.) Find the rank of B.

(b.) Find the dimension of the image of B.

(c.) Find the dimension of the kernel of B.

8 Linear independence, spanning set, basis for vectors

1.) Write down a linearly dependent spanning set of $\mathbb{R}^4$.

2.) Write down four linearly independent vectors in $\mathbb{R}^4$ which do not form a spanning set for $\mathbb{R}^4$.

3.) Write down three linearly independent vectors in $\mathbb{R}^4$ which do not form a spanning set for $\mathbb{R}^4$.

9 Linear independence, spanning set, basis for polynomials

1.) Write down a linearly dependent spanning set of $\mathbb{P}_3$.

2.) Write down four linearly independent vectors in $\mathbb{P}_3$ which do not form a spanning set for $\mathbb{P}_3$.

3.) Write down three linearly independent vectors in $\mathbb{P}_3$ which do not form a spanning set for $\mathbb{P}_3$.

10 Linear independence, spanning set, basis for other vector spaces

1.) Write down a set of two distinct objects in the vector space $\mathcal{C}^\infty$ of all smooth functions which are linearly independent.

2.) Does the set you wrote above forms a basis for $\mathcal{C}^\infty$? Why or why not?

3.) Describe the subspace of $\mathcal{C}^\infty$ which is spanned by the set you wrote above.

11 Lecture 16a vector space arithmetic

1.) Write $(x-4)^3$ as a scalar multiple of $1+x+x^2$ or state that it's impossible.

2.) Write x^2 as a linear combination of the polynomials in the set $\{1, 1+x, 1+2x+x^2\}$ or state that it's impossible.

12 Lecture 16b subspaces of a vector space

1.) Prove that $\{f(x) \in \mathbb{P}_2 \mid f(5) = 0\}$ is or is not a subspace.

2.) Prove that $\{f(x) \in \mathbb{P}_2 \mid f(5) = 1\}$ is or is not a subspace.

13 Lecture 17a linear independence and spanning set for polynomials

1.) Determine whether $\{x, x+1, x+2\}$ is linearly independent or linearly dependent.

2.) Determine whether $\{x, x+1, x+2\}$ is a spanning set for the vector space $\mathbb{P}_1$.

3.) Determine whether $\{x-1, x^2-1, x^2-2\}$ is linearly independent.

4.) Determine whether $\{x^2, (x-1)^2, (x-2)^2\}$ is a basis for the vector space $\mathbb{P}_2$.

14 Lecture 17b dimension and basis of a subspace

1.) Find the dimension of the subspace $\{f(x) \in \mathbb{P}_2 \mid f(5) = 0\}$ of $\mathbb{P}_2$.

2.) Find a basis for the subspace $\{f(x) \in \mathbb{P}_2 \mid f(5) = 0\}$ of $\mathbb{P}_2$.

15 Do you know how to use bases to convert elements in a general vector space into vectors?

(If not, review lecture17b.pdf starting from page 10 of the PDF file until the end.)

Compute the coefficient vector of x^2 in the basis $\{1, 1+x, 1+2x+x^2, x^3\}$ of $\mathbb{P}_3$.

16 Linear transformations for vectors

Let $F : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by =

$$F(p(x)) := (x-2)p'(x)$$

1.) Determine whether $x^2 - 1$ is in the kernel of F.

2.) Determine whether 16 is in the kernel of F.

3.) Determine whether $x^2 - 1$ is in the image of F.

4.) Determine whether $x^2 - 4$ is in the image of F.

5.) Find the dimension of the kernel of F. Write down a basis for the kernel of F.

6.) Find the dimension of the image of F. Write down a basis for the image of F.