Wk13 Wksheet (nine problems) **Math 3333**

References: Lecture 13a Bases, Lecture 13b Bases (Basis computations and dimensions), Lecture 14a Basis algorithm for the kernel of a matrix.

1 A basis of $\mathbb{R}^2$?

0.) Below, copy Theorem 2 and Theorem 3 on Slide 8/14 in lecture13a.pdf.

Determine if the following sets of vectors are a basis of the indicated space.

(Hint: See Exercise 5 in lecture13a.pdf.)

i.) Does the set of vectors $\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ form a basis of $\mathbb{R}^2$? Explain using theorems/computation/numbers and complete sentences.

ii.) Does the set of vectors $\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ form a basis of $\mathbb{R}^2$? Explain using theorems/computation/numbers and complete sentences.

iii.) Does the set of vectors $\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$ form a basis of $\mathbb{R}^2$? Explain using theorems/computation/numbers and complete sentences.

2 A basis of $\mathbb{R}^3$?

a.) Does the set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ form a basis of $\mathbb{R}^3$? Explain using theorems/computation/numbers and complete sentences.

b.) Does the set of vectors $\left\{ \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$ form a basis of $\mathbb{R}^3$? Explain using theorems/computation/numbers and complete sentences.

3 Find basis from the defining properties of sets

Find a basis and calculate the dimension of the following subspaces of $\mathbb{R}^4$.

a.) $V = \left\{ \begin{bmatrix} a-b \\ b+c \\ a \\ b+c \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$

b.) $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R}, a+b=c+d \right\}$

4 True or false?

For each statement, determine whether it is true or false. Justify your answer.

a.) If the columns of a 3×3 matrix A are a basis of $\mathbb{R}^3$, then the columns of A^T form a basis of $\mathbb{R}^3$.

(Hint: You can copy Exercise 4 in lecture 13a.pdf, but change the numbers accordingly.)

b.) If $\{\mathbf{x}, \mathbf{y}\}$ is a subset of $\mathbb{R}^3$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y}\}$ is linearly dependent.

c.) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly independent subset of $\mathbb{R}^3$, then $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

d.) If $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent subset of $\mathbb{R}^3$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

5 Basis of a Span, lecture13b.pdf

Hint: Review Algorithm 5 on slide 4/13 and Exercise 6 in lecture13b.pdf

a.) Find a basis of the subspace $U = \mathsf{span}\left\{ \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -10 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \right\}$ using Algorithm 5.

b.) Find the dimension of U.

6 Basis of an Image Space, lecture13b.pdf

Hint: Review Algorithm 5 (rephrased) on slide 5/13, Exercise 7(a), Theorem 6 on slide 7/13, and Exercise 7(b) in lecture13b.pdf.

a.) For the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 2 & 7 & -1 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix}$, find a basis of the subspace $\mathsf{im}(A)$ using Algorithm 5 (rephrased).

b.) Find the dimension of the subspace $\mathsf{im}(A)$.

7 Basis of Kernel, lecture14a.pdf

$$\text{Let } A := \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(a) Find a basis for $\mathsf{ker}(A)$ and the dimension of $\mathsf{ker}(A)$.

(b) Determine whether the following set of vectors is a basis for $\mathsf{ker}(A)$.

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

8 Basis of solution space to a HSLE, lecture14a.pdf

a.) Find a basis of the subspace of the solutions to the following homogeneous system of linear equations (HSLE):

$$\begin{aligned} x - 2y + z + w &= 0 \\ -x + 2y + 0z + w &= 0 \\ 2x - 4y + z + 0w &= 0 \end{aligned}$$

b.) What is the dimension of this subspace?

9 Basis of Eigenspace, lecture14a.pdf

a.) Find a basis of the 2-eigenspace of

$$M := \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

b.) What is the dimension of this eigenspace?