0. Optional Question (computing kernel, lecture12a.pdf), don't turn in

Let

$$B := \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}$$

Compute the kernel of B, that is, find *all* solutions to

$$B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note: For a similar problem, see Exercise 1 in lecture12a.pdf.

Your final answer should be equivalent to

$$\boxed{\ker B = \left\{ \begin{bmatrix} 2t+r \\ t \\ r \end{bmatrix} \text{ for } t, r \text{ in } \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ where } t, r \text{ are in } \mathbb{R} \right\}},$$

although your two vectors may be different.

1 Question (checking for spanning set, lecture12a.pdf)

Let V be the subspace of $\mathbb{R}^3$ consisting of height-3 vectors whose 3rd entry is the sum of the first two entries. For example, $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}4\\4\\8\end{bmatrix}$ are in V. (In the previous wksheet, we already showed that this subset is a subpsace.)

Let

$$S := \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}.$$

Show that S is a spanning set for V.

(For full credit, follow "SAMPLE STUDENT PROOF" for Exercise 3 of lecture12a.pdf, but we are working with a different subspace and different vectors, so your solution should use different matrices and computation.)

2 Question (checking for linear independence, lecture12b.pdf)

Let

$$S := \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(the same set as in the previous question).

a. Use the computation you did earlier to determine whether S is linearly independent or linearly dependent. Explain.

(Hint: For determining whether a set is linearly independent or linearly dependent, see lecture12b.pdf Exercise 5, 6, or 7.)

b. Let V be the subspace of $\mathbb{R}^3$ consisting of height-3 vectors whose 3rd entry is the sum of the first two entries (from the previous Question 1). Determine whether S is a basis for V. Give a brief explanation (write in complete sentences).

(Hint: Use both part (a) of this page and Question 1)

3 Question (kernel as linear combinations, a spanning set, and the image of a matrix, lecture12a.pdf)

$$\text{Let} \quad A := \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \end{bmatrix}.$$

a. Compute $\ker(A)$, i.e., find all solutions to $A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Parametrize the solution set *using parameters s and t.*

(Recommended sanity check: Pick a nonzero vector $\mathbf{v}$ in your solution set, compute $A\mathbf{v}$, and verify that it is equal to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

b. Find two vectors $\mathbf{v}$ and $\mathbf{w}$ where you can rewrite your parametrization of the kernel of A as $s\mathbf{v} + t\mathbf{w}$.

c. Finish each of the following sentences.

(i) The kernel of A is equal to *the set of all linear combinations* of the vectors ____________________

(ii) The kernel of A is equal to the *span* of the vectors ____________________

(iii) The kernel of A is equal to the *image* of a matrix ____________________
(Hint: See Exercise 2 & Fact 1 in lecture12a.pdf.)

4 Question

a. Let B be a 4×5 matrix. After row reducing to an REF matrix, we learn that this REF matrix has 3 leading 1s.

What is the rank of B?

How many parameters are required to parametrize the kernel of B?

How many vectors are required to span the kernel of B?

b. Let C be an $r \times c$ matrix and suppose C has rank k. How many vectors are required to parametrize the kernel of C?

How many vectors are required to span the kernel of C?

5 Challenge Problem (worth 0 points, but feedback will be given)

Show that, if the set of vectors

$$\{\mathbf{y}, \mathbf{v}, \mathbf{w}\}$$

is linearly independent, then the set of vectors

$$\{\mathbf{y} + \mathbf{v}, \mathbf{y} + \mathbf{w}\}$$

is also linearly independent.