1 Question (preserving addition + scalar multiplication, lecture10b.pdf)

Define $F : \mathbb{R}^4 \to \mathbb{R}$ by $F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = x_1 + x_2 + x_3 + x_4$.

We will now show that F is a linear transformation.

a.) First, show that F preserves addition.

b.) Next, show that F preserves scalar multiplication.

2 Question (standard basis vectors, lecture10b.pdf)

Define $F : \mathbb{R}^4 \to \mathbb{R}$ by $F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = x_1 + x_2 + x_3 + x_4$.

This is the same function as in the previous page.

a.) Write all the standard basis vectors of $\mathbb{R}^4$.

b.) Evaluate F at each standard basis vector.

c.) Use part (a) and (b) above to write a matrix A such that $F = T_A$.

3 Question (Geometric meaning of determinant, lecture10b.pdf.)

a.) If A is a 2×2 matrix, what does lecture10b.pdf say about the geometric meaning of $|\det(A)|$, the absolute value of the determinant of A? See the slide labeled 11/13.

b.) Suppose $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which reflects each point across a line through the origin.

What must $|\det(A)|$ be? **Your answer should be a specific number.**

(Hint: How does reflection change areas? Imagine looking in the mirror.)

4 Question (Geometric meaning of eigenvectors, lecture10b.pdf.)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix}.$$

a.) Find a matrix A such that f is equal to the linear transformation of A (denoted by T_A).

(Reference: Exercise 3 in lecture10b.pdf. Check your answer! Is $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ x+2y \end{bmatrix}$?)

b.) The matrix A has two eigenvalues, 1 and 3.

- The 1-eigenvectors of A are all nonzero vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

  In a previous wksheet, we showed that they are of the form $\begin{bmatrix} t \\ -t \end{bmatrix}$ for nonzero number t. For examples, $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 100 \\ -100 \end{bmatrix}$ are 1-eigenvectors of A.

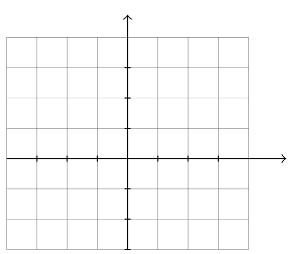
- The 3-eigenvectors of A are all nonzero vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $A\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$.

  In a previous wksheet, we showed that they are of the form $\begin{bmatrix} t \\ t \end{bmatrix}$ for nonzero number t. For examples, $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 100 \\ 100 \end{bmatrix}$ are 3-eigenvectors of A.

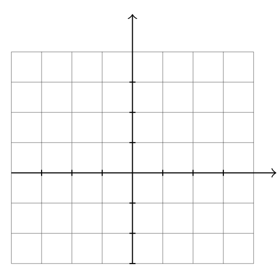
For each eigenvalue of A, pick one of the eigenvectors. Call your chosen 1-eigenvector $\mathbf{v}$ and your 3-eigenvector $\mathbf{w}$.

Your pick for a 1-eigenvector $\mathbf{v}$: ☐. Compute $f(\mathbf{v})$: ☐. Your 3-eigenvector $\mathbf{w}$: ☐. Compute $f(\mathbf{w})$: ☐.

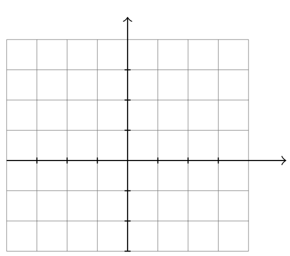
Plot $\mathbf{v}$ below



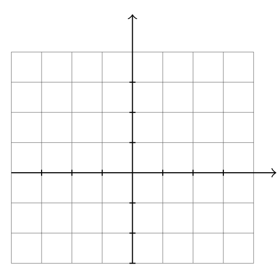
Plot $f(\mathbf{v})$ below



Plot $\mathbf{w}$ below



Plot $f(\mathbf{w})$ below



(Verify that f sends your v to a vector parallel to v, and that f sends your w to a vector parallel to w, as promised in *Eigenvectors revisited* in lecture10b.pdf)

5 Question (kernel of a matrix, lecture11a.pdf)

Let

$$B := \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}$$

a.) Write down the definition of the kernel of the 2×3 matrix B. by copying Slide 9/13 of lecture11a.pdf but replace m, n with actual numbers.)

b.) Using the method shown in Exercise 2 of lecture11a.pdf, determine whether $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is in the kernel of B. Show work.

c.) Using the method shown in Exercise 2 of lecture11a.pdf, determine whether $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the kernel of B. Show work.

6 Question (Image of a matrix, lecture11a.pdf)

Let

$$B := \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix} \quad \text{like in the previous page.}$$

a.) Write the definition of the image of B by copying Slide 12/13 of lecture11a.pdf but replace m, n with numbers.)

b.) Determine whether $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in the image of B. Show work, like in Exercise 3 of lecture11a.pdf.

c.) Determine whether $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is in the image of B. Show work, like in Exercise 3 of lecture11a.pdf

d.) Analyze the computation you did for $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then explain why every vector in the image of B must be of the form $\begin{bmatrix} t \\ -2t \end{bmatrix}$ for some number t.

(You don't need to write a proof, but explain your thought process using words and math symbols.)

7 Question (Copying sample student proof, lecture11b.pdf.)

(Warm-up: On scratch paper or a notebook or a note-taking app, copy the "SAMPLE STUDENT PROOF" from Exercise 6 given on page 15 of the file lecture11b.pdf. Do *not* submit this warm-up activity.)

Let S be the subset of vectors in $\mathbb{R}^2$ whose entries are the same. Show that S is a subspace of $\mathbb{R}^2$.

Hint: To earn full credit, you should simply copy the "SAMPLE STUDENT PROOF" from Exercise 6 of Lec 11b **but change the number and height of the vectors appropriately**.

8 Question (subspace proof, lecture11b.pdf.)

Hint: follow the sample answers from Exercises 6, 7 of lecture11b.pdf, but the set V here is completely different, so **don't simply copy**.

Let V be the subset of $\mathbb{R}^3$ consisting of height-3 vectors whose 3rd entry is the sum of the first two entries. (For example, $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}0\\2\\2\end{bmatrix}$, and $\begin{bmatrix}4\\4\\8\end{bmatrix}$ are in V). We will break down the steps of a proof that V is a subspace of $\mathbb{R}^3$.

a. Is V non-empty? Why and why not?

b. Let v and w be in V. What does this tell you about v and w?

c. Next, give an argument that $v + w$ is in V.

d. What have you just shown about V? (Hint: use the phrase "closed under ...")

e. We need another argument to complete the proof. Fill in the blanks below to complete the proof that V is a subspace.

Next, let v be in V and let r be in $\mathbb{R}$. This means we may write v as ...

We then check that

$rv =$

Therefore, rv is in V, so V is ________________________

We have shown that V is a subspace.

9 Challenge Problem (worth 0 points, but feedback will be given)

Show that, if W contains more than one element and W is a subspace of $\mathbb{R}$, then $W = \mathbb{R}$. (That is, the only two subspaces of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$.)