Wk06 Wksheet (10+1 problems) **name:** **class time:**

If you have a stylus and a tablet, you may export the PDF file to your favorite note-taking app. Otherwise, write on your own notebook paper. Put each question on its own page. (Some people also like to print the worksheet into paper and write directly on it.)

References: Lec 6a Determinants; 6b Determinants, second part; Lec 7a Cofactors (and how they relate to determinants and inverses)

1 Question: properties of determinant

a.) Suppose A and B are two square matrices of the same size. (This means det is defined for both of them.) According to property ii of determinant from Slide 3 of Lecture 6a notes, is $\det(A)\det(B)$ always equal to $\det(AB)$?

- ◯ Yes, for all A, B
- ◯ No, not necessarily

b.) According to property iii of determinant from Slide 6 of Lecture 6a notes, what is the determinant of the 6×6 identity matrix?

$$\det(Id) = \underline{\qquad}$$

c.) According to Slide 6 of Lecture 6a notes, if M is invertible and $\det(M^{-1}) = \frac{2}{5}$, what is $\det(M)$?

$$\det(M) = \underline{\qquad}$$

2 Question: applying properties of determinant

Suppose A and B are $n \times n$ matrices, and suppose A is invertible. Let $x := \det(A)$ and let $y := \det(B)$ Write the following determinants in terms of **only** x and y.

a.) $\det(A^4)$

b.) $\det(ABA^{-1})$

c.) $\det(ABA^{-1}BA)$

Reference for this page: lecture notes 6a.

3 Question: false or true?

(Scalar multiplication is defined in lecture notes 3a.)
If the statement is true, explain why. If the statement is false, give a counterexample.

<u>Statement</u>: **If M is a 3×3 matrix, then $\det(-4M)$ always equals $-4\det(M)$.**

4 Question: false or true?

(Matrix addition is defined in lecture notes 3a.)
If the statement is true, explain why. If the statement is false, give a counterexample.

<u>Statement</u>: **If A and B are 3×3 matrices, then $\det(A+B)$ always equals $\det(A) + \det(B)$.**

5 Question: true or false?

If the statement is true, explain why. If the statement is false, give a counterexample.

<u>Statement</u>: **If $\det(M) \neq 0$ and $MA = MB$, then $A = B$.**

6 Question: elementary row operations which change determinant

Perform row operations to turn the first determinant into a *multiple* of the last determinant, as explained in Lecture 6a notes, section: Computing det using row operations.

(To help you catch computation errors, write down each step of the row operations.)

a.) $\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{vmatrix}$

$$= \boxed{} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

b.) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \boxed{} \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

c.) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \boxed{} \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$

7 Question: rank, determinant, and invertibility

a.)
- Write down the theorem at the end of Lecture notes 5b which highlights the connection between invertibility and rank:

- Write down property i of the determinant given in Lecture notes 6a.

b.) Let M be a 5×5 matrix. If $\det(M) = 0$, list all possible values for the rank of M. Use part (a) to explain your answer. (Your explanation should be in complete sentence/s.)

c.) If you know the rank of a 5×5 matrix M, what can you say about the determinant of M? (No explanation needed for this part.)

(a) If the rank of M is 0, then $\det(M)$ is ______

(b) If the rank of M is 1, then $\det(M)$ is ______

(c) If the rank of M is 2, then $\det(M)$ is ______

(d) If the rank of M is 3, then $\det(M)$ is ______

(e) If the rank of M is 4, then $\det(M)$ is ______

(f) If the rank of M is 5, then $\det(M)$ is ______

8 Question: transpose and determinant

a.) What does Slide 6 of Lecture 6b notes say about the determinant of the transpose of a square matrix?

____

b.) Determine whether the statement is true or false.

Statement: **If M is a 4×4 matrix, then $\det(M^T M) \geq 0$.**

◯ True

◯ False

c.) If the statement is true, explain why. If the statement is false, give a counterexample.

9 Question: three methods to compute a 3×3 determinant

a.) Write down the equation for the line which passes through the points $(1, 3)$ and $(2, 5)$. (Then draw a sketch of the line on your scratch paper, for sanity check)).

b.) Let $B := \begin{bmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 2 & 5 & 1 \end{bmatrix}$. Compute $\det B$ **using all three methods**, carefully showing all steps of your work.

i. perform row operations until you reach an upper triangular matrix (Lecture 6a notes)

ii. Sarrus' rule (which *only* works for 3×3 matrices) (Lecture 6b notes)

iii. Cofactor expansion (Lecture 7a notes)

c.) Describe all (x, y) which satisfy the equation $\det \begin{bmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 2 & 5 & 1 \end{bmatrix} = 0$.

10 Question: determinant of a special matrix

Suppose a, b, c are three distinct numbers, and let $V := \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$.

a.) Compute $\det(V)$ using any of the three methods you have learned (or a combination of the three methods). For credit, carefully show each step of your calculation.

Simplify the expression as much as possible! (Hint: Use the fact that $x^2 - y^2 = (x-y)(x+y)$)

b.) If you haven't already done so above, write $\det(V)$ as a product.

(Hint: Use the fact that $x^2 - y^2 = (x-y)(x+y)$ in the above computation)

c.) Using the fact that a, b, and c are three distinct numbers, explain why $\det V$ must be a non-zero number. Write in complete sentences.

11 Challenge Problem (worth 0 points, will not influence course grade)

Let $A := \begin{bmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{bmatrix}$.

Show that if $abc \neq 0$ then A invertible, and find A^{-1} in this case.

To find A^{-1}, I suggest using the method shown in the end of Lecture 7a notes: Computing inverse using cofactors, starting from Slide 15.