Week 5 Worksheet (ten problems) **Math 3333**

If you have a stylus and a tablet, you may export the PDF file to your favorite note-taking app. Otherwise, write on your own notebook paper. Put each question on its own page. (Some people also like to print the worksheet into paper and write directly on it.)

References:

- Lecture 4b Matrix multiplication (properties of matrix arithmetic)
- Lecture 5a Matrix inverses;
- Lecture 5b Matrix inverses: algorithm

1 Question (Warning)

Statement: **If $MA = MB$ and M is not a zero matrix, then $A = B$.**

If the statement is true, explain why. If the statement is false, give a counterexample. (No credit is given for simply writing true or false.)

Hint: Review the end of Lecture notes 4b.

2 Question (Transpose)

Definition: A matrix M is called *symmetric* if M is equal to its transpose M^T.

Statement:

If each of A, B, and AB is a symmetric matrix, then A and B commute with each other.

If the statement is true, explain why. If the statement is false, give a counterexample. (No credit is given for simply writing true or false.)

Hint: Review how transpose interacts with matrix multiplication in lecture notes 4b, middle section.

3 Question (A non-square matrix cannot have an inverse)

Let

$$A := \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \qquad B := \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

i.) Compute AB and BA

ii.) Is B the inverse of A?

iii.) Does A have an inverse?

Hint: See Lecture 5a notes, Slide 9.

4 Question (proof by contradiction)

Let

$$A := \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}, \qquad B := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

i.) Compute BA. Leave the symbols a, b, c, d as they are.

ii.) Use your computation in part (i) to show that A has no inverse.

5 Question (Rearranging equations)

I have an invertible 3×3 matrix A, and I know that $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$.

Find a matrix B such that

$$BA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Hint: See Lecture 5a notes, Rearranging equations.

6 Question (Elementary row operations algorithm for inverses)

I was given 4×4 matrix M and I wished to compute its inverse (if it exists). I put my matrix M and the identity matrix side by side:

$$\left[\begin{array}{c|cccc} & 1 & 0 & 0 & 0 \\ \mathsf{M} & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}\right]$$

Then I performed a bunch of elementary row operations and got the following:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & -3 & 6 \\ 0 & 0 & 1 & 0 & 0 & 5 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 6 & 0 \end{array}\right]$$

i.) Write down the inverse of M if it exists, or state that M has no inverse.
ii.) Write a brief note for your future self explaining your answer.

Hint: See Lecture 5b notes, part: $n \times n$ matrix

7 Question (Elementary row operations algorithm for inverses)

Let

$$C := \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Perform the 'elementary row operations' algorithm explained in Lecture 5b by hand to either find the inverse of C or conclude that C is not invertible. **Show all steps and label each step clearly.**

8 Question (Solving a linear system using matrix inverse)

Solve the following linear system by using the inverse of the coefficient matrix, using the method explained in Lecture 5a notes, Exercise 5.

$$\begin{aligned} a + 2b + 0c + 0d &= 1 \\ 0a + 2b + 3c + 0d &= 0 \\ a \;\; + 3b + \;\; c + 0d &= 1 \\ 0a + 0b + 0c + \;\; d &= -9 \end{aligned}$$

(You can use the result from the previous problem. Don't redo the same computation!)

9 Question (2×2 determinant)

Let

$$\text{Let} \qquad A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a.) Write down the formula for the determinant $\det(A) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ following Lecture 5b notes, slide 3. (Leave the symbols a, b, c, d as they are.)

b.) Suppose the determinant of A is nonzero. Write down the inverse matrix A^{-1} following the formula in lecture 5b notes, slide 3.

c.) Use part (b) to write down the determinant of the matrix A^{-1}.

d.) Multiply the number you got in part (a) with the number you got in part (c). Is this product equal to 1?

☐ Yes, the product is equal to 1.

☐ No, the product is not equal to 1.

10 Question (rank and invertibility)

i.) Review the theorem "Invertibility and rank" in Lecture notes 5b.

Write down the theorem:

ii.) With no computation or a very little computation, determine whether the following matrices is invertible. Write a brief note to your future self (who will be preparing for an exam) explaining your reasoning.

Don't perform the algorithm for finding an inverse!

$$A := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \qquad B := \begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad C := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Hint: See Exercise 9 in Lecture notes 5b.