If you have a stylus and a tablet, you may export the PDF file to your favorite note-taking app. Otherwise, write on your own notebook paper. Put each question on its own page. (Some people also like to print the worksheet into paper and write directly on it.)

References: Lecture 3a Matrices and vectors (addition, scalar multiplication, linear combination, transpose); 3b Matrix and vector multiplication; 4a Matrix multiplication

1   Question (linear combination)

(a.) Write down the definition of what it means for a vector to be a *linear combination* of other vectors (see slide no. 11 of Lecture 3a).

(b.) Write the vector $\begin{bmatrix} -5 \\ 6 \end{bmatrix}$ as a concrete linear combination of the vectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Your scalars should be actual numbers written in Arabic numerals, not just variables.

2 Question (matrix and vector multiplication)

Compute the following product or state that it is undefined.

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 11 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

3 Question (matrix and vector multiplication)

Compute the following product or state that it is undefined.

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 11 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$
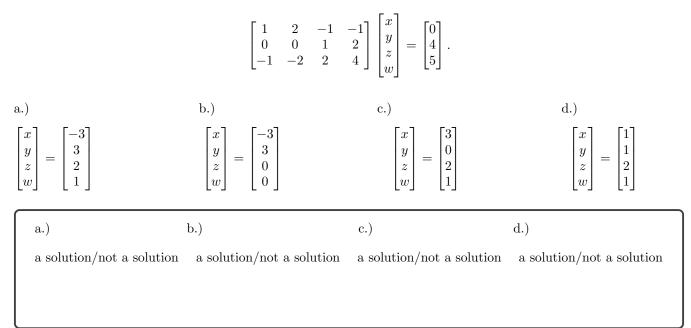
4   Question (transpose)

Recall from Lecture 3a the following properties of the transpose: Let A and B denote matrices of the same size, and let r be a number.

i. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

ii. $(A^T)^T = A$.

iii. $(rA)^T = rA^T$.

iv. $(A + B)^T = A^T + B^T$.

(a.) Copy Example 2.1.12 of the textbook.

(b.) Suppose that M is a 3×3 matrix which satisfies $M = 3M^T$. Solve for M. Mimic the above solution from the textbook example.

5   Question (matrix and vector multiplication)

For each of the four vectors below, perform a matrix-vector multiplication (like in Exercise 4 in Lecture 3b: egunawan.github.io/la/notes/lecture3b.pdf) to determine whether it is a solution to the matrix equation
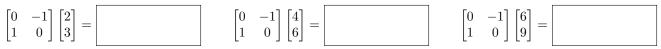
$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 \\ -1 & -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}.$$

a.)   b.)   c.)   d.)

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$ $$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$ $$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$ $$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

a.)   b.)   c.)   d.)

a solution/not a solution   a solution/not a solution   a solution/not a solution   a solution/not a solution

e.) Neatly write out your computation for at least one of (a)-(d) (the rest you can do on scratch paper).

6   Question (2D transformation)

Let $M := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

a. Compute the product $M \begin{bmatrix} x \\ y \end{bmatrix}$.

b. Compute the following vectors:

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} =$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} =$

c. Plot the points $v_1 = (2, 3)$, $v_2 = (4, 6)$, $v_3 = (6, 9)$ in Cartesian coordinates. On the same graph, plot the points corresponding to the vectors computed in part (b).

d. Describe what the matrix M does to the points v_1, v_2, v_3. Hint: Use phrases like "rotation by ... degrees" or "reflection across ... line".

e. What does the matrix M^2 do to the points v_1, v_2, v_3?

7　Question (2D transformation)

(This question is a 2D preview of a future topic.)

Let $B := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(a) Compute the product $B \begin{bmatrix} x \\ y \end{bmatrix}$.

(b) Compute the following vectors:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boxed{} \qquad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \boxed{} \qquad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \boxed{}$$

(c) Plot the points $v_1 = (2, 3)$, $v_2 = (4, 6)$, $v_3 = (6, 9)$ in Cartesian coordinates. On the same graph, plot the points corresponding to the vectors computed in part (b)

(d) Describe what the matrix B does to the points v_1, v_2, v_3. Hint: Use phrases like "rotation by ... degrees" or "reflection across ... line".

(e) What does the matrix B^2 do to the points v_1, v_2, v_3?

8   Question (Matrix multiplication)

Find all triples of numbers (x, y, z) so that $\begin{bmatrix} x & x \\ y & z \end{bmatrix}$ commutes with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. That is, find all (x, y, z) so that

$$\begin{bmatrix} x & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & x \\ y & z \end{bmatrix}.$$

(Explain the steps you use to get to your answers. Use full sentences and matrices.)

9 Question (Matrix multiplication)

Find all triples of numbers (a, b, c) so that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(Explain the steps you use to get to your answers. Use full sentences and matrices.)

10   Question (Review of past topic, Lecture 2a)

Recall that a 6×4 matrix has height 6 and width 4, meaning the matrix has 6 (horizontal) rows and 4 (vertical) columns.

a. Write down a non-zero 6×4 augmented matrix which is in row-echelon form (REF) and represents a linear system which has infinitely many solutions.

b. Write down a non-zero 6×4 augmented matrix which is in row-echelon form (REF) and represents a linear system which has exactly one solution, or explain why it's impossible.