# Chapter 3

## Determinants and Diagonalization

With each square matrix we can calculate a number, called the determinant of the matrix, which tells us whether or not the matrix is invertible. In fact, determinants can be used to give a formula for the inverse of a matrix. They also arise in calculating certain numbers (called eigenvalues) associated with the matrix. These eigenvalues are essential to a technique called diagonalization that is used in many applications where it is desired to predict the future behaviour of a system. For example, we use it to predict whether a species will become extinct.

Determinants were first studied by Leibnitz in 1696, and the term "determinant" was first used in 1801 by Gauss is his *Disquisitiones Arithmeticae*. Determinants are much older than matrices (which were introduced by Cayley in 1878) and were used extensively in the eighteenth and nineteenth centuries, primarily because of their significance in geometry (see Section 4.4). Although they are somewhat less important today, determinants still play a role in the theory and application of matrix algebra.

## 3.1 The Cofactor Expansion

In Section 2.4 we defined the determinant of a $2 \times 2$ matrix $A = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$ as follows:[1]

$$\det A = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| = ad - bc$$

and showed (in Example 2.4.4) that $A$ has an inverse if and only if $\det A \neq 0$. One objective of this chapter is to do this for *any* square matrix A. There is no difficulty for $1 \times 1$ matrices: If $A = [a]$, we define $\det A = \det [a] = a$ and note that $A$ is invertible if and only if $a \neq 0$.

If $A$ is $3 \times 3$ and invertible, we look for a suitable definition of $\det A$ by trying to carry $A$ to the identity matrix by row operations. The first column is not zero ($A$ is invertible); suppose the (1, 1)-entry $a$ is not zero. Then row operations give

$$A = \left[ \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \right] \rightarrow \left[ \begin{array}{ccc} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{array} \right] \rightarrow \left[ \begin{array}{ccc} a & b & c \\ 0 & ae-bd & af-cd \\ 0 & ah-bg & ai-cg \end{array} \right] = \left[ \begin{array}{ccc} a & b & c \\ 0 & u & af-cd \\ 0 & v & ai-cg \end{array} \right]$$

where $u = ae - bd$ and $v = ah - bg$. Since $A$ is invertible, one of $u$ and $v$ is nonzero (by Example 2.4.11); suppose that $u \neq 0$. Then the reduction proceeds

$$A \rightarrow \left[ \begin{array}{ccc} a & b & c \\ 0 & u & af-cd \\ 0 & v & ai-cg \end{array} \right] \rightarrow \left[ \begin{array}{ccc} a & b & c \\ 0 & u & af-cd \\ 0 & uv & u(ai-cg) \end{array} \right] \rightarrow \left[ \begin{array}{ccc} a & b & c \\ 0 & u & af-cd \\ 0 & 0 & w \end{array} \right]$$

where $w = u(ai - cg) - v(af - cd) = a(aei + bfg + cdh - ceg - afh - bdi)$. We define

$$\det A = aei + bfg + cdh - ceg - afh - bdi \tag{3.1}$$

[1]Determinants are commonly written $|A| = \det A$ using vertical bars. We will use both notations.

and observe that $\det A \neq 0$ because $a \det A = w \neq 0$ (is invertible).

To motivate the definition below, collect the terms in Equation 3.1 involving the entries $a$, $b$, and $c$ in row 1 of $A$:

$$\begin{aligned}\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= aei + bfg + cdh - ceg - afh - bdi \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}\end{aligned}$$

This last expression can be described as follows: To compute the determinant of a $3 \times 3$ matrix $A$, multiply each entry in row 1 by a sign times the determinant of the $2 \times 2$ matrix obtained by deleting the row and column of that entry, and add the results. The signs alternate down row 1, starting with $+$. It is this observation that we generalize below.

**Example 3.1.1**

$$\begin{aligned}\det \begin{bmatrix} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{bmatrix} &= 2\begin{vmatrix} 0 & 6 \\ 5 & 0 \end{vmatrix} - 3\begin{vmatrix} -4 & 6 \\ 1 & 0 \end{vmatrix} + 7\begin{vmatrix} -4 & 0 \\ 1 & 5 \end{vmatrix} \\ &= 2(-30) - 3(-6) + 7(-20) \\ &= -182\end{aligned}$$

This suggests an inductive method of defining the determinant of any square matrix in terms of determinants of matrices one size smaller. The idea is to define determinants of $3 \times 3$ matrices in terms of determinants of $2 \times 2$ matrices, then we do $4 \times 4$ matrices in terms of $3 \times 3$ matrices, and so on.

To describe this, we need some terminology.

**Definition 3.1 Cofactors of a Matrix**

*Assume that determinants of $(n-1) \times (n-1)$ matrices have been defined. Given the $n \times n$ matrix $A$, let*

*$A_{ij}$ denote the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting row $i$ and column $j$.*

*Then the $(i, j)$-**cofactor** $c_{ij}(A)$ is the scalar defined by*

$$c_{ij}(A) = (-1)^{i+j} \det (A_{ij})$$

*Here $(-1)^{i+j}$ is called the **sign** of the $(i, j)$-position.*

The sign of a position is clearly 1 or $-1$, and the following diagram is useful for remembering it:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Note that the signs alternate along each row and column with $+$ in the upper left corner.

**Example 3.1.2**

Find the cofactors of positions $(1, 2)$, $(3, 1)$, and $(2, 3)$ in the following matrix.

$$A = \begin{bmatrix} 3 & -1 & 6 \\ 5 & 2 & 7 \\ 8 & 9 & 4 \end{bmatrix}$$

**Solution.** Here $A_{12}$ is the matrix $\begin{bmatrix} 5 & 7 \\ 8 & 4 \end{bmatrix}$ that remains when row 1 and column 2 are deleted. The sign of position $(1, 2)$ is $(-1)^{1+2} = -1$ (this is also the $(1, 2)$-entry in the sign diagram), so the $(1, 2)$-cofactor is

$$c_{12}(A) = (-1)^{1+2} \begin{vmatrix} 5 & 7 \\ 8 & 4 \end{vmatrix} = (-1)(5 \cdot 4 - 7 \cdot 8) = (-1)(-36) = 36$$

Turning to position $(3, 1)$, we find

$$c_{31}(A) = (-1)^{3+1} A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 6 \\ 2 & 7 \end{vmatrix} = (+1)(-7 - 12) = -19$$

Finally, the $(2, 3)$-cofactor is

$$c_{23}(A) = (-1)^{2+3} A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & -1 \\ 8 & 9 \end{vmatrix} = (-1)(27 + 8) = -35$$

Clearly other cofactors can be found—there are nine in all, one for each position in the matrix.

We can now define $\det A$ for any square matrix $A$

**Definition 3.2 Cofactor expansion of a Matrix**

*Assume that determinants of* $(n-1) \times (n-1)$ *matrices have been defined. If* $A = \left[a_{ij}\right]$ *is* $n \times n$ *define*

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

*This is called the* ***cofactor expansion*** *of* $\det A$ *along row* 1.

It asserts that $\det A$ can be computed by multiplying the entries of row 1 by the corresponding cofactors, and adding the results. The astonishing thing is that $\det A$ can be computed by taking the cofactor expansion along *any row or column*: Simply multiply each entry of that row or column by the corresponding cofactor and add.

**Theorem 3.1.1: Cofactor Expansion Theorem**[2]

*The determinant of an $n \times n$ matrix $A$ can be computed by using the cofactor expansion along any row or column of $A$. That is $\det A$ can be computed by multiplying each entry of the row or column by the corresponding cofactor and adding the results.*

The proof will be given in Section 3.6.

**Example 3.1.3**

Compute the determinant of $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 7 & 2 \\ 9 & 8 & -6 \end{bmatrix}$.

**Solution.** The cofactor expansion along the first row is as follows:

$$\begin{aligned}\det A &= 3c_{11}(A) + 4c_{12}(A) + 5c_{13}(A) \\ &= 3\begin{vmatrix} 7 & 2 \\ 8 & -6 \end{vmatrix} - 4\begin{vmatrix} 1 & 2 \\ 9 & -6 \end{vmatrix} + 5\begin{vmatrix} 1 & 7 \\ 9 & 8 \end{vmatrix} \\ &= 3(-58) - 4(-24) + 5(-55) \\ &= -353\end{aligned}$$

Note that the signs alternate along the row (indeed along *any* row or column). Now we compute $\det A$ by expanding along the first column.

$$\begin{aligned}\det A &= 3c_{11}(A) + 1c_{21}(A) + 9c_{31}(A) \\ &= 3\begin{vmatrix} 7 & 2 \\ 8 & -6 \end{vmatrix} - \begin{vmatrix} 4 & 5 \\ 8 & -6 \end{vmatrix} + 9\begin{vmatrix} 4 & 5 \\ 7 & 2 \end{vmatrix} \\ &= 3(-58) - (-64) + 9(-27) \\ &= -353\end{aligned}$$

The reader is invited to verify that $\det A$ can be computed by expanding along any other row or column.

The fact that the cofactor expansion along *any row or column* of a matrix $A$ always gives the same result (the determinant of $A$) is remarkable, to say the least. The choice of a particular row or column can simplify the calculation.

[2]The cofactor expansion is due to Pierre Simon de Laplace (1749–1827), who discovered it in 1772 as part of a study of linear differential equations. Laplace is primarily remembered for his work in astronomy and applied mathematics.

**Example 3.1.4**

Compute $\det A$ where $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 5 & 1 & 2 & 0 \\ 2 & 6 & 0 & -1 \\ -6 & 3 & 1 & 0 \end{bmatrix}$.

**Solution.** The first choice we must make is which row or column to use in the cofactor expansion. The expansion involves multiplying entries by cofactors, so the work is minimized when the row or column contains as many zero entries as possible. Row 1 is a best choice in this matrix (column 4 would do as well), and the expansion is

$$\begin{aligned} \det A &= 3c_{11}(A) + 0c_{12}(A) + 0c_{13}(A) + 0c_{14}(A) \\ &= 3 \begin{vmatrix} 1 & 2 & 0 \\ 6 & 0 & -1 \\ 3 & 1 & 0 \end{vmatrix} \end{aligned}$$

This is the first stage of the calculation, and we have succeeded in expressing the determinant of the $4 \times 4$ matrix $A$ in terms of the determinant of a $3 \times 3$ matrix. The next stage involves this $3 \times 3$ matrix. Again, we can use any row or column for the cofactor expansion. The third column is preferred (with two zeros), so

$$\begin{aligned} \det A &= 3 \left( 0 \begin{vmatrix} 6 & 0 \\ 3 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ 6 & 0 \end{vmatrix} \right) \\ &= 3[0 + 1(-5) + 0] \\ &= -15 \end{aligned}$$

This completes the calculation.

Computing the determinant of a matrix $A$ can be tedious. For example, if $A$ is a $4 \times 4$ matrix, the cofactor expansion along any row or column involves calculating four cofactors, each of which involves the determinant of a $3 \times 3$ matrix. And if $A$ is $5 \times 5$, the expansion involves five determinants of $4 \times 4$ matrices! There is a clear need for some techniques to cut down the work.[3]

The motivation for the method is the observation (see Example 3.1.4) that calculating a determinant is simplified a great deal when a row or column consists mostly of zeros. (In fact, when a row or column consists *entirely* of zeros, the determinant is zero—simply expand along that row or column.)

Recall next that one method of *creating* zeros in a matrix is to apply elementary row operations to it. Hence, a natural question to ask is what effect such a row operation has on the determinant of the matrix. It turns out that the effect is easy to determine and that elementary *column* operations can be used in the same way. These observations lead to a technique for evaluating determinants that greatly reduces the

[3] If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ we can calculate $\det A$ by considering $\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$ obtained from $A$ by adjoining columns 1 and 2 on the right. Then $\det A = aei + bfg + cdh - ceg - afh - bdi$, where the positive terms $aei$, $bfg$, and $cdh$ are the products down and to the right starting at $a$, $b$, and $c$, and the negative terms $ceg$, $afh$, and $bdi$ are the products down and to the left starting at $c$, $a$, and $b$. **Warning**: This rule does **not** apply to $n \times n$ matrices where $n > 3$ or $n = 2$.

labour involved. The necessary information is given in Theorem 3.1.2.

**Theorem 3.1.2**

*Let $A$ denote an $n \times n$ matrix.*

1. *If $A$ has a row or column of zeros,* $\det A = 0$.
2. *If two distinct rows (or columns) of $A$ are interchanged, the determinant of the resulting matrix is* $-\det A$.
3. *If a row (or column) of $A$ is multiplied by a constant $u$, the determinant of the resulting matrix is* $u(\det A)$.
4. *If two distinct rows (or columns) of $A$ are identical,* $\det A = 0$.
5. *If a multiple of one row of $A$ is added to a different row (or if a multiple of a column is added to a different column), the determinant of the resulting matrix is* $\det A$.

**Proof.** We prove properties 2, 4, and 5 and leave the rest as exercises.

*Property 2.* If $A$ is $n \times n$, this follows by induction on $n$. If $n = 2$, the verification is left to the reader. If $n > 2$ and two rows are interchanged, let $B$ denote the resulting matrix. Expand $\det A$ and $\det B$ along a row *other than* the two that were interchanged. The entries in this row are the same for both $A$ and $B$, but the cofactors in $B$ are the negatives of those in $A$ (by induction) because the corresponding $(n-1) \times (n-1)$ matrices have two rows interchanged. Hence, $\det B = -\det A$, as required. A similar argument works if two columns are interchanged.

*Property 4.* If two rows of $A$ are equal, let $B$ be the matrix obtained by interchanging them. Then $B = A$, so $\det B = detA$. But $\det B = -\det A$ by property 2, so $\det A = \det B = 0$. Again, the same argument works for columns.

*Property 5.* Let $B$ be obtained from $A = \left[a_{ij}\right]$ by adding $u$ times row $p$ to row $q$. Then row $q$ of $B$ is

$$(a_{q1} + ua_{p1},\ a_{q2} + ua_{p2},\ \dots,\ a_{qn} + ua_{pn})$$

The cofactors of these elements in $B$ are the same as in $A$ (they do not involve row $q$): in symbols, $c_{qj}(B) = c_{qj}(A)$ for each $j$. Hence, expanding $B$ along row $q$ gives

$$\begin{aligned}
\det B &= (a_{q1} + ua_{p1})c_{q1}(A) + (a_{q2} + ua_{p2})c_{q2}(A) + \cdots + (a_{qn} + ua_{pn})c_{qn}(A) \\
&= [a_{q1}c_{q1}(A) + a_{q2}c_{q2}(A) + \cdots + a_{qn}c_{qn}(A)] + u[a_{p1}c_{q1}(A) + a_{p2}c_{q2}(A) + \cdots + a_{pn}c_{qn}(A)] \\
&= \det A + u \det C
\end{aligned}$$

where $C$ is the matrix obtained from $A$ by replacing row $q$ by row $p$ (and both expansions are along row $q$). Because rows $p$ and $q$ of $C$ are equal, $\det C = 0$ by property 4. Hence, $\det B = \det A$, as required. As before, a similar proof holds for columns. □

To illustrate Theorem 3.1.2, consider the following determinants.

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$ (because the last row consists of zeros)

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 8 & 7 \\ 1 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} 5 & -1 & 3 \\ 7 & 8 & 2 \\ -1 & 2 & 1 \end{vmatrix}$$ (because two columns are interchanged)

$$\begin{vmatrix} 8 & 1 & 2 \\ 3 & 0 & 9 \\ 1 & 2 & -1 \end{vmatrix} = 3\begin{vmatrix} 8 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & -1 \end{vmatrix}$$ (because the second row of the matrix on the left is 3 times the second row of the matrix on the right)

$$\begin{vmatrix} 2 & 1 & 2 \\ 4 & 0 & 4 \\ 1 & 3 & 1 \end{vmatrix} = 0$$ (because two columns are identical)

$$\begin{vmatrix} 2 & 5 & 2 \\ -1 & 2 & 9 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 9 & 20 \\ -1 & 2 & 9 \\ 3 & 1 & 1 \end{vmatrix}$$ (because twice the second row of the matrix on the left was added to the first row)

The following four examples illustrate how Theorem 3.1.2 is used to evaluate determinants.

**Example 3.1.5**

Evaluate $\det A$ when $A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & 6 \end{bmatrix}$.

**Solution.** The matrix does have zero entries, so expansion along (say) the second row would involve somewhat less work. However, a column operation can be used to get a zero in position (2, 3)—namely, add column 1 to column 3. Because this does not change the value of the determinant, we obtain

$$\det A = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 4 \\ 1 & 0 & 0 \\ 2 & 1 & 8 \end{vmatrix} = -\begin{vmatrix} -1 & 4 \\ 1 & 8 \end{vmatrix} = 12$$

where we expanded the second $3 \times 3$ matrix along row 2.

**Example 3.1.6**

If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 6$, evaluate $\det A$ where $A = \begin{bmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{bmatrix}$.

**Solution.** First take common factors out of rows 2 and 3.

$$\det A = 3(-1)\det \begin{bmatrix} a+x & b+y & c+z \\ x & y & z \\ p & q & r \end{bmatrix}$$

Now subtract the second row from the first and interchange the last two rows.

$$\det A = -3\det \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix} = 3\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 3\cdot 6 = 18$$

The determinant of a matrix is a sum of products of its entries. In particular, if these entries are polynomials in $x$, then the determinant itself is a polynomial in $x$. It is often of interest to determine which values of $x$ make the determinant zero, so it is very useful if the determinant is given in factored form. Theorem 3.1.2 can help.

**Example 3.1.7**

Find the values of $x$ for which $\det A = 0$, where $A = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$.

**Solution.** To evaluate $\det A$, first subtract $x$ times row 1 from rows 2 and 3.

$$\det A = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x \\ 0 & 1-x^2 & x-x^2 \\ 0 & x-x^2 & 1-x^2 \end{vmatrix} = \begin{vmatrix} 1-x^2 & x-x^2 \\ x-x^2 & 1-x^2 \end{vmatrix}$$

At this stage we could simply evaluate the determinant (the result is $2x^3-3x^2+1$). But then we would have to factor this polynomial to find the values of $x$ that make it zero. However, this factorization can be obtained directly by first factoring each entry in the determinant and taking a common factor of $(1-x)$ from each row.

$$\begin{aligned} \det A &= \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ x(1-x) & (1-x)(1+x) \end{vmatrix} = (1-x)^2 \begin{vmatrix} 1+x & x \\ x & 1+x \end{vmatrix} \\ &= (1-x)^2(2x+1) \end{aligned}$$

Hence, $\det A = 0$ means $(1-x)^2(2x+1) = 0$, that is $x = 1$ or $x = -\frac{1}{2}$.

**Example 3.1.8**

If $a_1$, $a_2$, and $a_3$ are given show that

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = (a_3 - a_1)(a_3 - a_2)(a_2 - a_1)$$

**Solution.** Begin by subtracting row 1 from rows 2 and 3, and then expand along column 1:

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 \\ 0 & a_3 - a_1 & a_3^2 - a_1^2 \end{bmatrix} = \det \begin{bmatrix} a_2 - a_1 & a_2^2 - a_1^2 \\ a_3 - a_1 & a_3^2 - a_1^2 \end{bmatrix}$$

Now $(a_2 - a_1)$ and $(a_3 - a_1)$ are common factors in rows 1 and 2, respectively, so

$$\begin{aligned} \det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} &= (a_2 - a_1)(a_3 - a_1) \det \begin{bmatrix} 1 & a_2 + a_1 \\ 1 & a_3 + a_1 \end{bmatrix} \\ &= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2) \end{aligned}$$

The matrix in Example 3.1.8 is called a Vandermonde matrix, and the formula for its determinant can be generalized to the $n \times n$ case (see Theorem 3.2.7).

If $A$ is an $n \times n$ matrix, forming $uA$ means multiplying *every* row of $A$ by $u$. Applying property 3 of Theorem 3.1.2, we can take the common factor $u$ out of each row and so obtain the following useful result.

**Theorem 3.1.3**

*If $A$ is an $n \times n$ matrix, then $\det(uA) = u^n \det A$ for any number $u$.*

The next example displays a type of matrix whose determinant is easy to compute.

**Example 3.1.9**

Evaluate $\det A$ if $A = \begin{bmatrix} a & 0 & 0 & 0 \\ u & b & 0 & 0 \\ v & w & c & 0 \\ x & y & z & d \end{bmatrix}$.

**Solution.** Expand along row 1 to get $\det A = a \begin{vmatrix} b & 0 & 0 \\ w & c & 0 \\ y & z & d \end{vmatrix}$. Now expand this along the top row to get $\det A = ab \begin{vmatrix} c & 0 \\ z & d \end{vmatrix} = abcd$, the product of the main diagonal entries.

A square matrix is called a **lower triangular matrix** if all entries above the main diagonal are zero (as in Example 3.1.9). Similarly, an **upper triangular matrix** is one for which all entries below the main diagonal are zero. A **triangular matrix** is one that is either upper or lower triangular. Theorem 3.1.4 gives an easy rule for calculating the determinant of any triangular matrix. The proof is like the solution to Example 3.1.9.

**Theorem 3.1.4**

*If $A$ is a square triangular matrix, then $\det A$ is the product of the entries on the main diagonal.*

Theorem 3.1.4 is useful in computer calculations because it is a routine matter to carry a matrix to triangular form using row operations.

Block matrices such as those in the next theorem arise frequently in practice, and the theorem gives an easy method for computing their determinants. This dovetails with Example 2.4.11.

**Theorem 3.1.5**

*Consider matrices $\left[\begin{array}{cc} A & X \\ 0 & B \end{array}\right]$ and $\left[\begin{array}{cc} A & 0 \\ Y & B \end{array}\right]$ in block form, where $A$ and $B$ are square matrices. Then*

$$\det \left[\begin{array}{cc} A & X \\ 0 & B \end{array}\right] = \det A \det B \text{ and } \det \left[\begin{array}{cc} A & 0 \\ Y & B \end{array}\right] = \det A \det B$$

**Proof.** Write $T = \det \left[\begin{array}{cc} A & X \\ 0 & B \end{array}\right]$ and proceed by induction on $k$ where $A$ is $k \times k$. If $k = 1$, it is the cofactor expansion along column 1. In general let $S_i(T)$ denote the matrix obtained from $T$ by deleting row $i$ and column 1. Then the cofactor expansion of $\det T$ along the first column is

$$\det T = a_{11} \det (S_1(T)) - a_{21} \det (S_2(T)) + \cdots \pm a_{k1} \det (S_k(T)) \tag{3.2}$$

where $a_{11},\ a_{21},\ \cdots,\ a_{k1}$ are the entries in the first column of $A$. But $S_i(T) = \left[\begin{array}{cc} S_i(A) & X_i \\ 0 & B \end{array}\right]$ for each $i = 1,\ 2,\ \cdots,\ k$, so $\det (S_i(T)) = \det (S_i(A)) \cdot \det B$ by induction. Hence, Equation 3.2 becomes

$$\begin{aligned} \det T &= \{a_{11} \det (S_1(T)) - a_{21} \det (S_2(T)) + \cdots \pm a_{k1} \det (S_k(T))\} \det B \\ &= \{\det A\} \det B \end{aligned}$$

as required. The lower triangular case is similar. □

**Example 3.1.10**

$$\det \left[\begin{array}{rrrr} 2 & 3 & 1 & 3 \\ 1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 1 \end{array}\right] = - \left|\begin{array}{rrrr} 2 & 1 & 3 & 3 \\ 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 \end{array}\right| = - \left|\begin{array}{rr} 2 & 1 \\ 1 & -1 \end{array}\right| \left|\begin{array}{rr} 1 & 1 \\ 4 & 1 \end{array}\right| = -(-3)(-3) = -9$$

The next result shows that $\det A$ is a linear transformation when regarded as a function of a fixed column of $A$. The proof is Exercise 3.1.21.

**Theorem 3.1.6**

*Given columns* $\mathbf{c}_1, \cdots, \mathbf{c}_{j-1}, \mathbf{c}_{j+1}, \cdots, \mathbf{c}_n$ *in* $\mathbb{R}^n$, *define* $T : \mathbb{R}^n \to \mathbb{R}$ *by*

$$T(\mathbf{x}) = \det \left[ \begin{array}{ccccccc} \mathbf{c}_1 & \cdots & \mathbf{c}_{j-1} & \mathbf{x} & \mathbf{c}_{j+1} & \cdots & \mathbf{c}_n \end{array} \right] \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

*Then, for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$ *and all* $a$ *in* $\mathbb{R}$,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \quad \text{and} \quad T(a\mathbf{x}) = aT(\mathbf{x})$$

## Exercises for 3.1

**Exercise 3.1.1** Compute the determinants of the following matrices.

a. $\left[ \begin{array}{rr} 2 & -1 \\ 3 & 2 \end{array} \right]$

b. $\left[ \begin{array}{rr} 6 & 9 \\ 8 & 12 \end{array} \right]$

c. $\left[ \begin{array}{cc} a^2 & ab \\ ab & b^2 \end{array} \right]$

d. $\left[ \begin{array}{cc} a+1 & a \\ a & a-1 \end{array} \right]$

e. $\left[ \begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right]$

f. $\left[ \begin{array}{rrr} 2 & 0 & -3 \\ 1 & 2 & 5 \\ 0 & 3 & 0 \end{array} \right]$

g. $\left[ \begin{array}{rrr} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right]$

h. $\left[ \begin{array}{ccc} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{array} \right]$

i. $\left[ \begin{array}{ccc} 1 & b & c \\ b & c & 1 \\ c & 1 & b \end{array} \right]$

j. $\left[ \begin{array}{ccc} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{array} \right]$

k. $\left[ \begin{array}{rrrr} 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \end{array} \right]$

l. $\left[ \begin{array}{rrrr} 1 & 0 & 3 & 1 \\ 2 & 2 & 6 & 0 \\ -1 & 0 & -3 & 1 \\ 4 & 1 & 12 & 0 \end{array} \right]$

m. $\left[ \begin{array}{rrrr} 3 & 1 & -5 & 2 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 5 & 2 \\ 1 & 1 & 2 & -1 \end{array} \right]$

n. $\left[ \begin{array}{rrrr} 4 & -1 & 3 & -1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & -1 & 1 \end{array} \right]$

o. $\left[ \begin{array}{rrrr} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{array} \right]$

p. $\left[ \begin{array}{cccc} 0 & 0 & 0 & a \\ 0 & 0 & b & p \\ 0 & c & q & k \\ d & s & t & u \end{array} \right]$

**Exercise 3.1.2** Show that $\det A = 0$ if $A$ has a row or column consisting of zeros.

**Exercise 3.1.3** Show that the sign of the position in the last row and the last column of $A$ is always $+1$.

**Exercise 3.1.4** Show that $\det I = 1$ for any identity matrix $I$.

**Exercise 3.1.5** Evaluate the determinant of each matrix by reducing it to upper triangular form.

a. $\left[ \begin{array}{rrr} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -1 & 3 \end{array} \right]$

b. $\left[ \begin{array}{rrr} -1 & 3 & 1 \\ 2 & 5 & 3 \\ 1 & -2 & 1 \end{array} \right]$

c. $\left[ \begin{array}{rrrr} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{array} \right]$

d. $\left[ \begin{array}{rrrr} 2 & 3 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 1 \\ 1 & 1 & 2 & 5 \end{array} \right]$

**Exercise 3.1.6** Evaluate by cursory inspection:

a. $\det \left[ \begin{array}{ccc} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{array} \right]$

b. $\det \left[ \begin{array}{ccc} a & b & c \\ a+b & 2b & c+b \\ 2 & 2 & 2 \end{array} \right]$

**Exercise 3.1.7** If $\det \left[ \begin{array}{ccc} a & b & c \\ p & q & r \\ x & y & z \end{array} \right] = -1$ compute:

a. $\det \left[ \begin{array}{ccc} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{array} \right]$

b. $\det \begin{bmatrix} -2a & -2b & -2c \\ 2p+x & 2q+y & 2r+z \\ 3x & 3y & 3z \end{bmatrix}$

**Exercise 3.1.8** Show that:

a. $\det \begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} = 2 \det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

b. $\det \begin{bmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{bmatrix} = 9 \det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

**Exercise 3.1.9** In each case either prove the statement or give an example showing that it is false:

a. $\det(A+B) = \det A + \det B$.

b. If $\det A = 0$, then $A$ has two equal rows.

c. If $A$ is $2 \times 2$, then $\det(A^T) = \det A$.

d. If $R$ is the reduced row-echelon form of $A$, then $\det A = \det R$.

e. If $A$ is $2 \times 2$, then $\det(7A) = 49 \det A$.

f. $\det(A^T) = -\det A$.

g. $\det(-A) = -\det A$.

h. If $\det A = \det B$ where $A$ and $B$ are the same size, then $A = B$.

**Exercise 3.1.10** Compute the determinant of each matrix, using Theorem 3.1.5.

a. $\begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ -1 & 3 & 1 & 4 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 3 & 0 & 1 \end{bmatrix}$

**Exercise 3.1.11** If $\det A = 2$, $\det B = -1$, and $\det C = 3$, find:

a. $\det \begin{bmatrix} A & X & Y \\ 0 & B & Z \\ 0 & 0 & C \end{bmatrix}$

b. $\det \begin{bmatrix} A & 0 & 0 \\ X & B & 0 \\ Y & Z & C \end{bmatrix}$

c. $\det \begin{bmatrix} A & X & Y \\ 0 & B & 0 \\ 0 & Z & C \end{bmatrix}$

d. $\det \begin{bmatrix} A & X & 0 \\ 0 & B & 0 \\ Y & Z & C \end{bmatrix}$

**Exercise 3.1.12** If $A$ has three columns with only the top two entries nonzero, show that $\det A = 0$.

**Exercise 3.1.13**

a. Find $\det A$ if $A$ is $3 \times 3$ and $\det(2A) = 6$.

b. Under what conditions is $\det(-A) = \det A$?

**Exercise 3.1.14** Evaluate by first adding all other rows to the first row.

a. $\det \begin{bmatrix} x-1 & 2 & 3 \\ 2 & -3 & x-2 \\ -2 & x & -2 \end{bmatrix}$

b. $\det \begin{bmatrix} x-1 & -3 & 1 \\ 2 & -1 & x-1 \\ -3 & x+2 & -2 \end{bmatrix}$

**Exercise 3.1.15**

a. Find $b$ if $\det \begin{bmatrix} 5 & -1 & x \\ 2 & 6 & y \\ -5 & 4 & z \end{bmatrix} = ax + by + cz$.

b. Find $c$ if $\det \begin{bmatrix} 2 & x & -1 \\ 1 & y & 3 \\ -3 & z & 4 \end{bmatrix} = ax + by + cz$.

**Exercise 3.1.16** Find the real numbers $x$ and $y$ such that $\det A = 0$ if:

a. $A = \begin{bmatrix} 0 & x & y \\ y & 0 & x \\ x & y & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & x & x \\ -x & -2 & x \\ -x & -x & -3 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{bmatrix}$

d. $A = \begin{bmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ y & 0 & 0 & x \end{bmatrix}$

**Exercise 3.1.17** Show that

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & x \\ 1 & x & 0 & x \\ 1 & x & x & 0 \end{bmatrix} = -3x^2$$

**Exercise 3.1.18** Show that

$$\det \begin{bmatrix} 1 & x & x^2 & x^3 \\ a & 1 & x & x^2 \\ p & b & 1 & x \\ q & r & c & 1 \end{bmatrix} = (1-ax)(1-bx)(1-cx).$$

**Exercise 3.1.19**

Given the polynomial $p(x) = a + bx + cx^2 + dx^3 + x^4$, the matrix $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & -c & -d \end{bmatrix}$ is called the **companion matrix** of $p(x)$. Show that $\det(xI - C) = p(x)$.

**Exercise 3.1.20** Show that

$$\det \begin{bmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{bmatrix}$$
$$= (a+b+c+3x)[(ab+ac+bc) - (a^2+b^2+c^2)]$$

**Exercise 3.1.21** . Prove Theorem 3.1.6. [*Hint*: Expand the determinant along column $j$.]

**Exercise 3.1.22** Show that

$$\det \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_2 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1} & \cdots & * & * \\ a_n & * & \cdots & * & * \end{bmatrix} = (-1)^k a_1 a_2 \cdots a_n$$

where either $n = 2k$ or $n = 2k+1$, and $*$-entries are arbitrary.

**Exercise 3.1.23** By expanding along the first column, show that:

$$\det \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = 1 + (-1)^{n+1}$$

if the matrix is $n \times n$, $n \geq 2$.

**Exercise 3.1.24** Form matrix $B$ from a matrix $A$ by writing the columns of $A$ in reverse order. Express $\det B$ in terms of $\det A$.

**Exercise 3.1.25** Prove property 3 of Theorem 3.1.2 by expanding along the row (or column) in question.

**Exercise 3.1.26** Show that the line through two distinct points $(x_1, y_1)$ and $(x_2, y_2)$ in the plane has equation

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

**Exercise 3.1.27** Let $A$ be an $n \times n$ matrix. Given a polynomial $p(x) = a_0 + a_1x + \cdots + a_mx^m$, we write $p(A) = a_0I + a_1A + \cdots + a_mA^m$.

For example, if $p(x) = 2 - 3x + 5x^2$, then $p(A) = 2I - 3A + 5A^2$. The *characteristic polynomial* of $A$ is defined to be $c_A(x) = \det[xI - A]$, and the Cayley-Hamilton theorem asserts that $c_A(A) = 0$ for any matrix $A$.

a. Verify the theorem for

i. $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$

ii. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 8 & 2 & 2 \end{bmatrix}$

b. Prove the theorem for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

## 3.2 Determinants and Matrix Inverses

In this section, several theorems about determinants are derived. One consequence of these theorems is that a square matrix $A$ is invertible if and only if $\det A \neq 0$. Moreover, determinants are used to give a formula for $A^{-1}$ which, in turn, yields a formula (called Cramer's rule) for the solution of any system of linear equations with an invertible coefficient matrix.

We begin with a remarkable theorem (due to Cauchy in 1812) about the determinant of a product of matrices. The proof is given at the end of this section.

**Theorem 3.2.1: Product Theorem**

*If $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = \det A \det B$.*

The complexity of matrix multiplication makes the product theorem quite unexpected. Here is an example where it reveals an important numerical identity.

**Example 3.2.1**

If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$ then $AB = \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{bmatrix}$.

Hence $\det A \det B = \det(AB)$ gives the identity

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$

Theorem 3.2.1 extends easily to $\det(ABC) = \det A \det B \det C$. In fact, induction gives

$$\det(A_1A_2\cdots A_{k-1}A_k) = \det A_1 \det A_2 \cdots \det A_{k-1} \det A_k$$

for any square matrices $A_1, \dots, A_k$ of the same size. In particular, if each $A_i = A$, we obtain

$$\det(A^k) = (detA)^k, \quad \text{for any } k \geq 1$$

We can now give the invertibility condition.

**Theorem 3.2.2**

*An $n \times n$ matrix $A$ is invertible if and only if $\det A \neq 0$. When this is the case, $\det(A^{-1}) = \frac{1}{\det A}$*

**Proof.** If $A$ is invertible, then $AA^{-1} = I$; so the product theorem gives

$$1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}$$

Hence, $\det A \neq 0$ and also $\det A^{-1} = \frac{1}{\det A}$.

Conversely, if $\det A \neq 0$, we show that $A$ can be carried to $I$ by elementary row operations (and invoke Theorem 2.4.5). Certainly, $A$ can be carried to its reduced row-echelon form $R$, so $R = E_k \cdots E_2E_1A$ where the $E_i$ are elementary matrices (Theorem 2.5.1). Hence the product theorem gives

$$\det R = \det E_k \cdots \det E_2 \det E_1 \det A$$

Since $\det E \neq 0$ for all elementary matrices $E$, this shows $\det R \neq 0$. In particular, $R$ has no row of zeros, so $R = I$ because $R$ is square and reduced row-echelon. This is what we wanted. □

**Example 3.2.2**

For which values of $c$ does $A = \begin{bmatrix} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{bmatrix}$ have an inverse?

**Solution.** Compute $\det A$ by first adding $c$ times column 1 to column 3 and then expanding along row 1.

$$\det A = \det \begin{bmatrix} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 1-c \\ 0 & 2c & -4 \end{bmatrix} = 2(c+2)(c-3)$$

Hence, $\det A = 0$ if $c = -2$ or $c = 3$, and $A$ has an inverse if $c \neq -2$ and $c \neq 3$.

**Example 3.2.3**

If a product $A_1 A_2 \cdots A_k$ of square matrices is invertible, show that each $A_i$ is invertible.

**Solution.** We have $\det A_1 \det A_2 \cdots \det A_k = \det (A_1 A_2 \cdots A_k)$ by the product theorem, and $\det (A_1 A_2 \cdots A_k) \neq 0$ by Theorem 3.2.2 because $A_1 A_2 \cdots A_k$ is invertible. Hence

$$\det A_1 \det A_2 \cdots \det A_k \neq 0$$

so $\det A_i \neq 0$ for each $i$. This shows that each $A_i$ is invertible, again by Theorem 3.2.2.

**Theorem 3.2.3**

*If $A$ is any square matrix,* $\det A^T = \det A$.

**Proof.** Consider first the case of an elementary matrix $E$. If $E$ is of type I or II, then $E^T = E$; so certainly $\det E^T = \det E$. If $E$ is of type III, then $E^T$ is also of type III; so $\det E^T = 1 = \det E$ by Theorem 3.1.2. Hence, $\det E^T = \det E$ for every elementary matrix $E$.

Now let $A$ be any square matrix. If $A$ is not invertible, then neither is $A^T$; so $\det A^T = 0 = \det A$ by Theorem 3.2.2. On the other hand, if $A$ is invertible, then $A = E_k \cdots E_2 E_1$, where the $E_i$ are elementary matrices (Theorem 2.5.2). Hence, $A^T = E_1^T E_2^T \cdots E_k^T$ so the product theorem gives

$$\begin{aligned}\det A^T = \det E_1^T \det E_2^T \cdots \det E_k^T &= \det E_1 \det E_2 \cdots \det E_k \\ &= \det E_k \cdots \det E_2 \det E_1 \\ &= \det A\end{aligned}$$

This completes the proof. □

**Example 3.2.4**

If $\det A = 2$ and $\det B = 5$, calculate $\det(A^3 B^{-1} A^T B^2)$.

**Solution.** We use several of the facts just derived.

$$\begin{aligned}\det(A^3 B^{-1} A^T B^2) &= \det(A^3) \det(B^{-1}) \det(A^T) \det(B^2) \\ &= (\det A)^3 \frac{1}{\det B} \det A (\det B)^2 \\ &= 2^3 \cdot \tfrac{1}{5} \cdot 2 \cdot 5^2 \\ &= 80\end{aligned}$$

**Example 3.2.5**

A square matrix is called **orthogonal** if $A^{-1} = A^T$. What are the possible values of $\det A$ if $A$ is orthogonal?

**Solution.** If $A$ is orthogonal, we have $I = AA^T$. Take determinants to obtain

$$1 = \det I = \det(AA^T) = \det A \det A^T = (\det A)^2$$

Since $\det A$ is a number, this means $\det A = \pm 1$.

Hence Theorems 2.6.4 and 2.6.5 imply that rotation about the origin and reflection about a line through the origin in $\mathbb{R}^2$ have orthogonal matrices with determinants 1 and $-1$ respectively. In fact they are the *only* such transformations of $\mathbb{R}^2$. We have more to say about this in Section 8.2.

## Adjugates

In Section 2.4 we defined the adjugate of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be $\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then we verified that $A(\operatorname{adj} A) = (\det A)I = (\operatorname{adj} A)A$ and hence that, if $\det A \neq 0$, $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$. We are now able to define the adjugate of an arbitrary square matrix and to show that this formula for the inverse remains valid (when the inverse exists).

Recall that the $(i, j)$-cofactor $c_{ij}(A)$ of a square matrix $A$ is a number defined for each position $(i, j)$ in the matrix. If $A$ is a square matrix, the **cofactor matrix of** $A$ is defined to be the matrix $\left[c_{ij}(A)\right]$ whose $(i, j)$-entry is the $(i, j)$-cofactor of $A$.

**Definition 3.3 Adjugate of a Matrix**

*The* **adjugate**[4] *of* $A$, *denoted* $\operatorname{adj}(A)$, *is the transpose of this cofactor matrix; in symbols,*

$$\operatorname{adj}(A) = \left[c_{ij}(A)\right]^T$$

This agrees with the earlier definition for a $2 \times 2$ matrix $A$ as the reader can verify.

**Example 3.2.6**

Compute the adjugate of $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix}$ and calculate $A(\operatorname{adj} A)$ and $(\operatorname{adj} A)A$.

**Solution.** We first find the cofactor matrix.

$$\begin{bmatrix} c_{11}(A) & c_{12}(A) & c_{13}(A) \\ c_{21}(A) & c_{22}(A) & c_{23}(A) \\ c_{31}(A) & c_{32}(A) & c_{33}(A) \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 1 & 5 \\ -6 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ -2 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & -6 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ -6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}$$

Then the adjugate of $A$ is the transpose of this cofactor matrix.

$$\operatorname{adj} A = \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}^T = \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

The computation of $A(\operatorname{adj} A)$ gives

$$A(\operatorname{adj} A) = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix} \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I$$

and the reader can verify that also $(\operatorname{adj} A)A = 3I$. Hence, analogy with the $2 \times 2$ case would indicate that $\det A = 3$; this is, in fact, the case.

The relationship $A(\operatorname{adj} A) = (\det A)I$ holds for any square matrix $A$. To see why this is so, consider

[4]This is also called the classical adjoint of $A$, but the term "adjoint" has another meaning.

the general $3 \times 3$ case. Writing $c_{ij}(A) = c_{ij}$ for short, we have

$$\text{adj } A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

If $A = \left[a_{ij}\right]$ in the usual notation, we are to verify that $A(\text{adj } A) = (\det A)I$. That is,

$$A(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

Consider the $(1, 1)$-entry in the product. It is given by $a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$, and this is just the cofactor expansion of $\det A$ along the first row of $A$. Similarly, the $(2, 2)$-entry and the $(3, 3)$-entry are the cofactor expansions of $\det A$ along rows 2 and 3, respectively.

So it remains to be seen why the off-diagonal elements in the matrix product $A(\text{adj } A)$ are all zero. Consider the $(1, 2)$-entry of the product. It is given by $a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23}$. This *looks* like the cofactor expansion of the determinant of *some* matrix. To see which, observe that $c_{21}$, $c_{22}$, and $c_{23}$ are all computed by *deleting* row 2 of $A$ (and one of the columns), so they remain the same if row 2 of $A$ is changed. In particular, if row 2 of $A$ is replaced by row 1, we obtain

$$a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0$$

where the expansion is along row 2 and where the determinant is zero because two rows are identical. A similar argument shows that the other off-diagonal entries are zero.

This argument works in general and yields the first part of Theorem 3.2.4. The second assertion follows from the first by multiplying through by the scalar $\frac{1}{\det A}$.

**Theorem 3.2.4: Adjugate Formula**

*If A is any square matrix, then*

$$A(\text{adj } A) = (\det A)I = (\text{adj } A)A$$

*In particular, if det A $\neq$ 0, the inverse of A is given by*

$$A^{-1} = \frac{1}{\det A} \text{adj } A$$

It is important to note that this theorem is *not* an efficient way to find the inverse of the matrix $A$. For example, if $A$ were $10 \times 10$, the calculation of $\text{adj } A$ would require computing $10^2 = 100$ determinants of $9 \times 9$ matrices! On the other hand, the matrix inversion algorithm would find $A^{-1}$ with about the same effort as finding $\det A$. Clearly, Theorem 3.2.4 is not a *practical* result: its virtue is that it gives a formula for $A^{-1}$ that is useful for *theoretical* purposes.

**Example 3.2.7**

Find the $(2, 3)$-entry of $A^{-1}$ if $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

**Solution.** First compute

$$\det A = \begin{vmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 7 \\ 5 & -7 & 11 \\ 3 & 0 & 0 \end{vmatrix} = 3 \begin{vmatrix} 1 & 7 \\ -7 & 11 \end{vmatrix} = 180$$

Since $A^{-1} = \frac{1}{\det A} \text{adj } A = \frac{1}{180} \left[ c_{ij}(A) \right]^T$, the $(2, 3)$-entry of $A^{-1}$ is the $(3, 2)$-entry of the matrix $\frac{1}{180} \left[ c_{ij}(A) \right]$; that is, it equals $\frac{1}{180} c_{32}(A) = \frac{1}{180} \left( - \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} \right) = \frac{13}{180}$.

**Example 3.2.8**

If $A$ is $n \times n$, $n \geq 2$, show that $\det(\text{adj } A) = (\det A)^{n-1}$.

**Solution.** Write $d = \det A$; we must show that $\det(\text{adj } A) = d^{n-1}$. We have $A(\text{adj } A) = dI$ by Theorem 3.2.4, so taking determinants gives $d \det(\text{adj } A) = d^n$. Hence we are done if $d \neq 0$. Assume $d = 0$; we must show that $\det(\text{adj } A) = 0$, that is, $\text{adj } A$ is not invertible. If $A \neq 0$, this follows from $A(\text{adj } A) = dI = 0$; if $A = 0$, it follows because then $\text{adj } A = 0$.

## Cramer's Rule

Theorem 3.2.4 has a nice application to linear equations. Suppose

$$A\mathbf{x} = \mathbf{b}$$

is a system of $n$ equations in $n$ variables $x_1, x_2, \dots, x_n$. Here $A$ is the $n \times n$ coefficient matrix, and $\mathbf{x}$ and $\mathbf{b}$ are the columns

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

of variables and constants, respectively. If $\det A \neq 0$, we left multiply by $A^{-1}$ to obtain the solution $\mathbf{x} = A^{-1}\mathbf{b}$. When we use the adjugate formula, this becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A} (\text{adj } A)\mathbf{b}$$

$$= \frac{1}{\det A} \begin{bmatrix} c_{11}(A) & c_{21}(A) & \cdots & c_{n1}(A) \\ c_{12}(A) & c_{22}(A) & \cdots & c_{n2}(A) \\ \vdots & \vdots & & \vdots \\ c_{1n}(A) & c_{2n}(A) & \cdots & c_{nn}(A) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Hence, the variables $x_1, x_2, \dots, x_n$ are given by

$$\begin{aligned} x_1 &= \tfrac{1}{\det A} \left[ b_1 c_{11}(A) + b_2 c_{21}(A) + \cdots + b_n c_{n1}(A) \right] \\ x_2 &= \tfrac{1}{\det A} \left[ b_1 c_{12}(A) + b_2 c_{22}(A) + \cdots + b_n c_{n2}(A) \right] \\ &\vdots \qquad\qquad \vdots \\ x_n &= \tfrac{1}{\det A} \left[ b_1 c_{1n}(A) + b_2 c_{2n}(A) + \cdots + b_n c_{nn}(A) \right] \end{aligned}$$

Now the quantity $b_1 c_{11}(A) + b_2 c_{21}(A) + \cdots + b_n c_{n1}(A)$ occurring in the formula for $x_1$ looks like the cofactor expansion of the determinant of a matrix. The cofactors involved are $c_{11}(A), c_{21}(A), \dots, c_{n1}(A)$, corresponding to the first column of $A$. If $A_1$ is obtained from $A$ by replacing the first column of $A$ by $\mathbf{b}$, then $c_{i1}(A_1) = c_{i1}(A)$ for each $i$ because column 1 is deleted when computing them. Hence, expanding $\det(A_1)$ by the first column gives

$$\begin{aligned} \det A_1 &= b_1 c_{11}(A_1) + b_2 c_{21}(A_1) + \cdots + b_n c_{n1}(A_1) \\ &= b_1 c_{11}(A) + b_2 c_{21}(A) + \cdots + b_n c_{n1}(A) \\ &= (\det A) x_1 \end{aligned}$$

Hence, $x_1 = \frac{\det A_1}{\det A}$ and similar results hold for the other variables.

**Theorem 3.2.5: Cramer's Rule[5]**

*If $A$ is an invertible $n \times n$ matrix, the solution to the system*

$$A\mathbf{x} = \mathbf{b}$$

*of $n$ equations in the variables $x_1, x_2, \dots, x_n$ is given by*

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \cdots, \quad x_n = \frac{\det A_n}{\det A}$$

*where, for each $k$, $A_k$ is the matrix obtained from $A$ by replacing column $k$ by* $\mathbf{b}$.

**Example 3.2.9**

Find $x_1$, given the following system of equations.

$$\begin{aligned} 5x_1 + x_2 - \phantom{5}x_3 &= 4 \\ 9x_1 + x_2 - \phantom{5}x_3 &= 1 \\ x_1 - x_2 + 5x_3 &= 2 \end{aligned}$$

[5]Gabriel Cramer (1704–1752) was a Swiss mathematician who wrote an introductory work on algebraic curves. He popularized the rule that bears his name, but the idea was known earlier.

**Solution.** Compute the determinants of the coefficient matrix $A$ and the matrix $A_1$ obtained from it by replacing the first column by the column of constants.

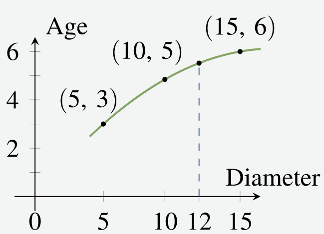
$$\det A = \det \begin{bmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{bmatrix} = -16$$
$$\det A_1 = \det \begin{bmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{bmatrix} = 12$$

Hence, $x_1 = \frac{\det A_1}{\det A} = -\frac{3}{4}$ by Cramer's rule.

Cramer's rule is *not* an efficient way to solve linear systems or invert matrices. True, it enabled us to calculate $x_1$ here without computing $x_2$ or $x_3$. Although this might seem an advantage, the truth of the matter is that, for large systems of equations, the number of computations needed to find *all* the variables by the gaussian algorithm is comparable to the number required to find *one* of the determinants involved in Cramer's rule. Furthermore, the algorithm works when the matrix of the system is not invertible and even when the coefficient matrix is not square. Like the adjugate formula, then, Cramer's rule is *not* a practical numerical technique; its virtue is theoretical.

## Polynomial Interpolation

**Example 3.2.10**




A forester wants to estimate the age (in years) of a tree by measuring the diameter of the trunk (in cm). She obtains the following data:

| | Tree 1 | Tree 2 | Tree 3 |
|---|---|---|---|
| Trunk Diameter | 5 | 10 | 15 |
| Age | 3 | 5 | 6 |

Estimate the age of a tree with a trunk diameter of 12 cm.

**Solution.**
The forester decides to "fit" a quadratic polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2$$

to the data, that is choose the coefficients $r_0$, $r_1$, and $r_2$ so that $p(5) = 3$, $p(10) = 5$, and $p(15) = 6$, and then use $p(12)$ as the estimate. These conditions give three linear equations:

$$\begin{aligned} r_0 + \phantom{1}5r_1 + \phantom{1}25r_2 &= 3 \\ r_0 + 10r_1 + 100r_2 &= 5 \\ r_0 + 15r_1 + 225r_2 &= 6 \end{aligned}$$

The (unique) solution is $r_0 = 0$, $r_1 = \frac{7}{10}$, and $r_2 = -\frac{1}{50}$ , so

$$p(x) = \frac{7}{10}x - \frac{1}{50}x^2 = \frac{1}{50}x(35 - x)$$

Hence the estimate is $p(12) = 5.52$.

As in Example 3.2.10, it often happens that two variables $x$ and $y$ are related but the actual functional form $y = f(x)$ of the relationship is unknown. Suppose that for certain values $x_1, x_2, \dots, x_n$ of $x$ the corresponding values $y_1, y_2, \dots, y_n$ are known (say from experimental measurements). One way to estimate the value of $y$ corresponding to some other value $a$ of $x$ is to find a polynomial[6]

$$p(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{n-1}x^{n-1}$$

that "fits" the data, that is $p(x_i) = y_i$ holds for each $i = 1, 2, \dots, n$. Then the estimate for $y$ is $p(a)$. As we will see, such a polynomial always exists if the $x_i$ are distinct.

The conditions that $p(x_i) = y_i$ are

$$\begin{array}{ccccccccc} r_0 + r_1x_1 + r_2x_1^2 + \cdots + r_{n-1}x_1^{n-1} = y_1 \\ r_0 + r_1x_2 + r_2x_2^2 + \cdots + r_{n-1}x_2^{n-1} = y_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ r_0 + r_1x_n + r_2x_n^2 + \cdots + r_{n-1}x_n^{n-1} = y_n \end{array}$$

In matrix form, this is

$$\left[\begin{array}{ccccc} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{array}\right] \left[\begin{array}{c} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{array}\right] = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array}\right] \tag{3.3}$$

It can be shown (see Theorem 3.2.7) that the determinant of the coefficient matrix equals the product of all terms $(x_i - x_j)$ with $i > j$ and so is nonzero (because the $x_i$ are distinct). Hence the equations have a unique solution $r_0, r_1, \dots, r_{n-1}$. This proves

**Theorem 3.2.6**

*Let $n$ data pairs $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ be given, and assume that the $x_i$ are distinct. Then there exists a unique polynomial*

$$p(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{n-1}x^{n-1}$$

*such that $p(x_i) = y_i$ for each $i = 1, 2, \dots, n$.*

The polynomial in Theorem 3.2.6 is called the **interpolating polynomial** for the data.

We conclude by evaluating the determinant of the coefficient matrix in Equation 3.3. If $a_1, a_2, \dots, a_n$ are numbers, the determinant

$$\det \left[\begin{array}{ccccc} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{array}\right]$$

[6]A **polynomial** is an expression of the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where the $a_i$ are numbers and $x$ is a variable. If $a_n \neq 0$, the integer $n$ is called the degree of the polynomial, and $a_n$ is called the leading coefficient. See Appendix D.

is called a **Vandermonde determinant**.[7] There is a simple formula for this determinant. If $n=2$, it equals $(a_2-a_1)$; if $n=3$, it is $(a_3-a_2)(a_3-a_1)(a_2-a_1)$ by Example 3.1.8. The general result is the product

$$\prod_{1\le j<i\le n}(a_i-a_j)$$

of all factors $(a_i-a_j)$ where $1\le j<i\le n$. For example, if $n=4$, it is

$$(a_4-a_3)(a_4-a_2)(a_4-a_1)(a_3-a_2)(a_3-a_1)(a_2-a_1)$$

**Theorem 3.2.7**

*Let $a_1,\ a_2,\ \dots,\ a_n$ be numbers where $n\ge 2$. Then the corresponding Vandermonde determinant is given by*

$$\det\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} = \prod_{1\le j<i\le n}(a_i-a_j)$$

Proof. We may assume that the $a_i$ are distinct; otherwise both sides are zero. We proceed by induction on $n\ge 2$; we have it for $n=2,\ 3$. So assume it holds for $n-1$. The trick is to replace $a_n$ by a variable $x$, and consider the determinant

$$p(x)=\det\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix}$$

Then $p(x)$ is a polynomial of degree at most $n-1$ (expand along the last row), and $p(a_i)=0$ for each $i=1,\ 2,\ \dots,\ n-1$ because in each case there are two identical rows in the determinant. In particular, $p(a_1)=0$, so we have $p(x)=(x-a_1)p_1(x)$ by the factor theorem (see Appendix D). Since $a_2\ne a_1$, we obtain $p_1(a_2)=0$, and so $p_1(x)=(x-a_2)p_2(x)$. Thus $p(x)=(x-a_1)(x-a_2)p_2(x)$. As the $a_i$ are distinct, this process continues to obtain

$$p(x)=(x-a_1)(x-a_2)\cdots(x-a_{n-1})d \tag{3.4}$$

where $d$ is the coefficient of $x^{n-1}$ in $p(x)$. By the cofactor expansion of $p(x)$ along the last row we get

$$d=(-1)^{n+n}\det\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-2} \end{bmatrix}$$

[7]Alexandre Théophile Vandermonde (1735–1796) was a French mathematician who made contributions to the theory of equations.

Because $(-1)^{n+n} = 1$ the induction hypothesis shows that $d$ is the product of all factors $(a_i - a_j)$ where $1 \leq j < i \leq n-1$. The result now follows from Equation 3.4 by substituting $a_n$ for $x$ in $p(x)$. □

**Proof of Theorem 3.2.1.** If $A$ and $B$ are $n \times n$ matrices we must show that

$$\det(AB) = \det A \det B \tag{3.5}$$

Recall that if $E$ is an elementary matrix obtained by doing one row operation to $I_n$, then doing that operation to a matrix $C$ (Lemma 2.5.1) results in $EC$. By looking at the three types of elementary matrices separately, Theorem 3.1.2 shows that

$$\det(EC) = \det E \det C \quad \text{for any matrix } C \tag{3.6}$$

Thus if $E_1, E_2, \dots, E_k$ are all elementary matrices, it follows by induction that

$$\det(E_k \cdots E_2 E_1 C) = \det E_k \cdots \det E_2 \det E_1 \det C \text{ for any matrix } C \tag{3.7}$$

*Lemma.* If $A$ has no inverse, then $\det A = 0$.

*Proof*. Let $A \to R$ where $R$ is reduced row-echelon, say $E_n \cdots E_2 E_1 A = R$. Then $R$ has a row of zeros by Part (4) of Theorem 2.4.5, and hence $\det R = 0$. But then Equation 3.7 gives $\det A = 0$ because $\det E \neq 0$ for any elementary matrix $E$. This proves the Lemma.

Now we can prove Equation 3.5 by considering two cases.

*Case 1. A has no inverse*. Then $AB$ also has no inverse (otherwise $A[B(AB)^{-1}] = I$ so $A$ is invertible by Corollary 2.4.1 to Theorem 2.4.5). Hence the above Lemma (twice) gives

$$\det(AB) = 0 = 0 \det B = \det A \det B$$

proving Equation 3.5 in this case.

*Case 2. A has an inverse.* Then $A$ is a product of elementary matrices by Theorem 2.5.2, say $A = E_1 E_2 \cdots E_k$. Then Equation 3.7 with $C = I$ gives

$$\det A = \det(E_1 E_2 \cdots E_k) = \det E_1 \det E_2 \cdots \det E_k$$

But then Equation 3.7 with $C = B$ gives

$$\det(AB) = \det[(E_1 E_2 \cdots E_k)B] = \det E_1 \det E_2 \cdots \det E_k \det B = \det A \det B$$

and Equation 3.5 holds in this case too. □

# Exercises for 3.2

**Exercise 3.2.1** Find the adjugate of each of the following matrices.

a. $\begin{bmatrix} 5 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

d. $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

**Exercise 3.2.2** Use determinants to find which real values of $c$ make each of the following matrices invertible.

a. $\begin{bmatrix} 1 & 0 & 3 \\ 3 & -4 & c \\ 2 & 5 & 8 \end{bmatrix}$ b. $\begin{bmatrix} 0 & c & -c \\ -1 & 2 & 1 \\ c & -c & c \end{bmatrix}$

c. $\begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ d. $\begin{bmatrix} 4 & c & 3 \\ c & 2 & c \\ 5 & c & 4 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{bmatrix}$ f. $\begin{bmatrix} 1 & c & -1 \\ c & 1 & 1 \\ 0 & 1 & c \end{bmatrix}$

**Exercise 3.2.3** Let $A$, $B$, and $C$ denote $n \times n$ matrices and assume that $\det A = -1$, $\det B = 2$, and $\det C = 3$. Evaluate:

a. $\det(A^3BC^TB^{-1})$ b. $\det(B^2C^{-1}AB^{-1}C^T)$

**Exercise 3.2.4** Let $A$ and $B$ be invertible $n \times n$ matrices. Evaluate:

a. $\det(B^{-1}AB)$ b. $\det(A^{-1}B^{-1}AB)$

**Exercise 3.2.5** If $A$ is $3 \times 3$ and $\det(2A^{-1}) = -4$ and $\det(A^3(B^{-1})^T) = -4$, find $\det A$ and $\det B$.

**Exercise 3.2.6** Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$ and assume that $\det A = 3$. Compute:

a. $\det(2B^{-1})$ where $B = \begin{bmatrix} 4u & 2a & -p \\ 4v & 2b & -q \\ 4w & 2c & -r \end{bmatrix}$

b. $\det(2C^{-1})$ where $C = \begin{bmatrix} 2p & -a+u & 3u \\ 2q & -b+v & 3v \\ 2r & -c+w & 3w \end{bmatrix}$

**Exercise 3.2.7** If $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -2$ calculate:

a. $\det \begin{bmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{bmatrix}$

b. $\det \begin{bmatrix} 2b & 0 & 4d \\ 1 & 2 & -2 \\ a+1 & 2 & 2(c-1) \end{bmatrix}$

c. $\det(3A^{-1})$ where $A = \begin{bmatrix} 3c & a+c \\ 3d & b+d \end{bmatrix}$

**Exercise 3.2.8** Solve each of the following by Cramer's rule:

a. $\begin{aligned} 2x + y &= 1 \\ 3x + 7y &= -2 \end{aligned}$ b. $\begin{aligned} 3x + 4y &= 9 \\ 2x - y &= -1 \end{aligned}$

c. $\begin{aligned} 5x + y - z &= -7 \\ 2x - y - 2z &= 6 \\ 3x + 2z &= -7 \end{aligned}$ d. $\begin{aligned} 4x - y + 3z &= 1 \\ 6x + 2y - z &= 0 \\ 3x + 3y + 2z &= -1 \end{aligned}$

**Exercise 3.2.9** Use Theorem 3.2.4 to find the (2, 3)-entry of $A^{-1}$ if:

a. $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 0 & 4 & 7 \end{bmatrix}$

**Exercise 3.2.10** Explain what can be said about $\det A$ if:

a. $A^2 = A$
b. $A^2 = I$
c. $A^3 = A$
d. $PA = P$ and $P$ is invertible
e. $A^2 = uA$ and $A$ is $n \times n$
f. $A = -A^T$ and $A$ is $n \times n$
g. $A^2 + I = 0$ and $A$ is $n \times n$

**Exercise 3.2.11** Let $A$ be $n \times n$. Show that $uA = (uI)A$, and use this with Theorem 3.2.1 to deduce the result in Theorem 3.1.3: $\det(uA) = u^n \det A$.

**Exercise 3.2.12** If $A$ and $B$ are $n \times n$ matrices, if $AB = -BA$, and if $n$ is odd, show that either $A$ or $B$ has no inverse.

**Exercise 3.2.13** Show that $\det AB = \det BA$ holds for any two $n \times n$ matrices $A$ and $B$.

**Exercise 3.2.14** If $A^k = 0$ for some $k \geq 1$, show that $A$ is not invertible.

**Exercise 3.2.15** If $A^{-1} = A^T$, describe the cofactor matrix of $A$ in terms of $A$.

**Exercise 3.2.16** Show that no $3 \times 3$ matrix $A$ exists such that $A^2 + I = 0$. Find a $2 \times 2$ matrix $A$ with this property.

**Exercise 3.2.17** Show that $\det(A + B^T) = \det(A^T + B)$ for any $n \times n$ matrices $A$ and $B$.

**Exercise 3.2.18** Let $A$ and $B$ be invertible $n \times n$ matrices. Show that $\det A = \det B$ if and only if $A = UB$ where $U$ is a matrix with $\det U = 1$.

**Exercise 3.2.19** For each of the matrices in Exercise 2, find the inverse for those values of $c$ for which it exists.

**Exercise 3.2.20** In each case either prove the statement or give an example showing that it is false:

a. If $\operatorname{adj} A$ exists, then $A$ is invertible.

b. If $A$ is invertible and $\operatorname{adj} A = A^{-1}$, then $\det A = 1$.

c. $\det(AB) = \det(B^T A)$.

d. If $\det A \neq 0$ and $AB = AC$, then $B = C$.

e. If $A^T = -A$, then $\det A = -1$.

f. If $\operatorname{adj} A = 0$, then $A = 0$.

g. If $A$ is invertible, then $\operatorname{adj} A$ is invertible.

h. If $A$ has a row of zeros, so also does $\operatorname{adj} A$.

i. $\det(A^T A) > 0$ for all square matrices $A$.

j. $\det(I + A) = 1 + \det A$.

k. If $AB$ is invertible, then $A$ and $B$ are invertible.

l. If $\det A = 1$, then $\operatorname{adj} A = A$.

m. If $A$ is invertible and $\det A = d$, then $\operatorname{adj} A = dA^{-1}$.

**Exercise 3.2.21** If $A$ is $2 \times 2$ and $\det A = 0$, show that one column of $A$ is a scalar multiple of the other. [*Hint*: Definition 2.5 and Part (2) of Theorem 2.4.5.]

**Exercise 3.2.22** Find a polynomial $p(x)$ of degree 2 such that:

a. $p(0) = 2$, $p(1) = 3$, $p(3) = 8$

b. $p(0) = 5$, $p(1) = 3$, $p(2) = 5$

**Exercise 3.2.23** Find a polynomial $p(x)$ of degree 3 such that:

a. $p(0) = p(1) = 1$, $p(-1) = 4$, $p(2) = -5$

b. $p(0) = p(1) = 1$, $p(-1) = 2$, $p(-2) = -3$

**Exercise 3.2.24** Given the following data pairs, find the interpolating polynomial of degree at most 3 and estimate the value of $y$ corresponding to $x = 1.5$.

a. $(0,\ 1), (1,\ 2), (2,\ 5), (3,\ 10)$

b. $(0,\ 1), (1,\ 1.49), (2,\ -0.42), (3,\ -11.33)$

c. $(0,\ 2), (1,\ 2.03), (2,\ -0.40), (-1,\ 0.89)$

**Exercise 3.2.25** If $A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ show that $\det A = 1 + a^2 + b^2 + c^2$. Hence, find $A^{-1}$ for any $a$, $b$, and $c$.

**Exercise 3.2.26**

a. Show that $A = \begin{bmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{bmatrix}$ has an inverse if and only if $abc \neq 0$, and find $A^{-1}$ in that case.

b. Show that if an upper triangular matrix is invertible, the inverse is also upper triangular.

**Exercise 3.2.27** Let $A$ be a matrix each of whose entries are integers. Show that each of the following conditions implies the other.

1. $A$ is invertible and $A^{-1}$ has integer entries.

2. $\det A = 1$ or $-1$.

**Exercise 3.2.28** If $A^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$ find $\operatorname{adj} A$.

**Exercise 3.2.29** If $A$ is $3 \times 3$ and $\det A = 2$, find $\det(A^{-1} + 4 \operatorname{adj} A)$.

**Exercise 3.2.30** Show that $\det \begin{bmatrix} 0 & A \\ B & X \end{bmatrix} = \det A \det B$ when $A$ and $B$ are $2 \times 2$. What if $A$ and $B$ are $3 \times 3$?

[*Hint*: Block multiply by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.]

**Exercise 3.2.31** Let $A$ be $n \times n$, $n \geq 2$, and assume one column of $A$ consists of zeros. Find the possible values of $\operatorname{rank}(\operatorname{adj} A)$.

**Exercise 3.2.32** If $A$ is $3 \times 3$ and invertible, compute $\det(-A^2(\operatorname{adj} A)^{-1})$.

**Exercise 3.2.33** Show that $\text{adj}\,(uA) = u^{n-1}\,\text{adj}\,A$ for all $n \times n$ matrices $A$.

**Exercise 3.2.34** Let $A$ and $B$ denote invertible $n \times n$ matrices. Show that:

a. $\text{adj}\,(\text{adj}\,A) = (\det A)^{n-2}A$ (here $n \geq 2$) [*Hint*: See Example 3.2.8.]

b. $\text{adj}\,(A^{-1}) = (\text{adj}\,A)^{-1}$

c. $\text{adj}\,(A^T) = (\text{adj}\,A)^T$

d. $\text{adj}\,(AB) = (\text{adj}\,B)(\text{adj}\,A)$ [*Hint*: Show that $AB\,\text{adj}\,(AB) = AB\,\text{adj}\,B\,\text{adj}\,A$.]

## 3.3 Diagonalization and Eigenvalues

The world is filled with examples of systems that evolve in time—the weather in a region, the economy of a nation, the diversity of an ecosystem, etc. Describing such systems is difficult in general and various methods have been developed in special cases. In this section we describe one such method, called *diagonalization,* which is one of the most important techniques in linear algebra. A very fertile example of this procedure is in modelling the growth of the population of an animal species. This has attracted more attention in recent years with the ever increasing awareness that many species are endangered. To motivate the technique, we begin by setting up a simple model of a bird population in which we make assumptions about survival and reproduction rates.

**Example 3.3.1**

Consider the evolution of the population of a species of birds. Because the number of males and females are nearly equal, we count only females. We assume that each female remains a juvenile for one year and then becomes an adult, and that only adults have offspring. We make three assumptions about reproduction and survival rates:

1. The number of juvenile females hatched in any year is twice the number of adult females alive the year before (we say the **reproduction rate** is 2).
2. Half of the adult females in any year survive to the next year (the **adult survival rate** is $\frac{1}{2}$).
3. One quarter of the juvenile females in any year survive into adulthood (the **juvenile survival rate** is $\frac{1}{4}$).

If there were 100 adult females and 40 juvenile females alive initially, compute the population of females $k$ years later.

**Solution.** Let $a_k$ and $j_k$ denote, respectively, the number of adult and juvenile females after $k$ years, so that the total female population is the sum $a_k + j_k$. Assumption 1 shows that $j_{k+1} = 2a_k$, while assumptions 2 and 3 show that $a_{k+1} = \frac{1}{2}a_k + \frac{1}{4}j_k$. Hence the numbers $a_k$ and $j_k$ in successive years are related by the following equations:

$$\begin{aligned} a_{k+1} &= \tfrac{1}{2}a_k + \tfrac{1}{4}j_k \\ j_{k+1} &= 2a_k \end{aligned}$$

If we write $\mathbf{v}_k = \left[\begin{array}{c} a_k \\ j_k \end{array}\right]$ and $A = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{array}\right]$ these equations take the matrix form

$$\mathbf{v}_{k+1} = A\mathbf{v}_k, \quad \text{for each } k = 0,\ 1,\ 2,\ \dots$$

Taking $k=0$ gives $\mathbf{v}_1 = A\mathbf{v}_0$, then taking $k=1$ gives $\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$, and taking $k=2$ gives $\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$. Continuing in this way, we get

$$\mathbf{v}_k = A^k\mathbf{v}_0, \quad \text{for each } k = 0,\ 1,\ 2,\ \dots$$

Since $\mathbf{v}_0 = \left[\begin{array}{c} a_0 \\ j_0 \end{array}\right] = \left[\begin{array}{c} 100 \\ 40 \end{array}\right]$ is known, finding the population profile $\mathbf{v}_k$ amounts to computing $A^k$ for all $k \geq 0$. We will complete this calculation in Example 3.3.12 after some new techniques have been developed.

Let $A$ be a fixed $n \times n$ matrix. A sequence $\mathbf{v}_0,\ \mathbf{v}_1,\ \mathbf{v}_2,\ \dots$ of column vectors in $\mathbb{R}^n$ is called a **linear dynamical system**[8] if $\mathbf{v}_0$ is known and the other $\mathbf{v}_k$ are determined (as in Example 3.3.1) by the conditions

$$\mathbf{v}_{k+1} = A\mathbf{v}_k \text{ for each } k = 0,\ 1,\ 2,\ \dots$$

These conditions are called a **matrix recurrence** for the vectors $\mathbf{v}_k$. As in Example 3.3.1, they imply that

$$\mathbf{v}_k = A^k\mathbf{v}_0 \text{ for all } k \geq 0$$

so finding the columns $\mathbf{v}_k$ amounts to calculating $A^k$ for $k \geq 0$.

Direct computation of the powers $A^k$ of a square matrix $A$ can be time-consuming, so we adopt an indirect method that is commonly used. The idea is to first **diagonalize** the matrix $A$, that is, to find an invertible matrix $P$ such that

$$P^{-1}AP = D \text{ is a diagonal matrix} \tag{3.8}$$

This works because the powers $D^k$ of the diagonal matrix $D$ are easy to compute, and Equation 3.8 enables us to compute powers $A^k$ of the matrix $A$ in terms of powers $D^k$ of $D$. Indeed, we can solve Equation 3.8 for $A$ to get $A = PDP^{-1}$. Squaring this gives

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

Using this we can compute $A^3$ as follows:

$$A^3 = AA^2 = (PDP^{-1})(PD^2P^{-1}) = PD^3P^{-1}$$

Continuing in this way we obtain Theorem 3.3.1 (even if $D$ is not diagonal).

**Theorem 3.3.1**

*If $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$ for each $k = 1,\ 2,\ \dots$.*

Hence computing $A^k$ comes down to finding an invertible matrix $P$ as in equation Equation 3.8. To do this it is necessary to first compute certain numbers (called eigenvalues) associated with the matrix $A$.

[8]More precisely, this is a linear discrete dynamical system. Many models regard $\mathbf{v}_t$ as a continuous function of the time $t$, and replace our condition between $\mathbf{b}_{k+1}$ and $A\mathbf{v}_k$ with a differential relationship viewed as functions of time.

## Eigenvalues and Eigenvectors

**Definition 3.4 Eigenvalues and Eigenvectors of a Matrix**

*If $A$ is an $n \times n$ matrix, a number $\lambda$ is called an **eigenvalue** of $A$ if*

$$A\mathbf{x} = \lambda\mathbf{x} \text{ for some column } \mathbf{x} \neq \mathbf{0} \text{ in } \mathbb{R}^n$$

*In this case, $\mathbf{x}$ is called an **eigenvector** of $A$ corresponding to the eigenvalue $\lambda$, or a $\lambda$-**eigenvector** for short.*

**Example 3.3.2**

If $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ then $A\mathbf{x} = 4\mathbf{x}$ so $\lambda = 4$ is an eigenvalue of $A$ with corresponding eigenvector $\mathbf{x}$.

The matrix $A$ in Example 3.3.2 has another eigenvalue in addition to $\lambda = 4$. To find it, we develop a general procedure for *any* $n \times n$ matrix $A$.

By definition a number $\lambda$ is an eigenvalue of the $n \times n$ matrix $A$ if and only if $A\mathbf{x} = \lambda\mathbf{x}$ for some column $\mathbf{x} \neq \mathbf{0}$. This is equivalent to asking that the homogeneous system

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$. By Theorem 2.4.5 this happens if and only if the matrix $\lambda I - A$ is not invertible and this, in turn, holds if and only if the determinant of the coefficient matrix is zero:

$$\det(\lambda I - A) = 0$$

This last condition prompts the following definition:

**Definition 3.5 Characteristic Polynomial of a Matrix**

*If $A$ is an $n \times n$ matrix, the **characteristic polynomial** $c_A(x)$ of $A$ is defined by*

$$c_A(x) = \det(xI - A)$$

Note that $c_A(x)$ is indeed a polynomial in the variable $x$, and it has degree $n$ when $A$ is an $n \times n$ matrix (this is illustrated in the examples below). The above discussion shows that a number $\lambda$ is an eigenvalue of $A$ if and only if $c_A(\lambda) = 0$, that is if and only if $\lambda$ is a **root** of the characteristic polynomial $c_A(x)$. We record these observations in

**Theorem 3.3.2**

*Let $A$ be an $n \times n$ matrix.*

1. *The eigenvalues $\lambda$ of $A$ are the roots of the characteristic polynomial $c_A(x)$ of $A$.*

2. *The $\lambda$-eigenvectors* $\mathbf{x}$ *are the nonzero solutions to the homogeneous system*

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

*of linear equations with $\lambda I - A$ as coefficient matrix.*

In practice, solving the equations in part 2 of Theorem 3.3.2 is a routine application of gaussian elimination, but finding the eigenvalues can be difficult, often requiring computers (see Section 8.5). For now, the examples and exercises will be constructed so that the roots of the characteristic polynomials are relatively easy to find (usually integers). However, the reader should not be misled by this into thinking that eigenvalues are so easily obtained for the matrices that occur in practical applications!

**Example 3.3.3**

Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ discussed in Example 3.3.2, and then find all the eigenvalues and their eigenvectors.

**Solution.** Since $xI - A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x-3 & -5 \\ -1 & x+1 \end{bmatrix}$ we get

$$c_A(x) = \det \begin{bmatrix} x-3 & -5 \\ -1 & x+1 \end{bmatrix} = x^2 - 2x - 8 = (x-4)(x+2)$$

Hence, the roots of $c_A(x)$ are $\lambda_1 = 4$ and $\lambda_2 = -2$, so these are the eigenvalues of $A$. Note that $\lambda_1 = 4$ was the eigenvalue mentioned in Example 3.3.2, but we have found a new one: $\lambda_2 = -2$. To find the eigenvectors corresponding to $\lambda_2 = -2$, observe that in this case

$$(\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} \lambda_2 - 3 & -5 \\ -1 & \lambda_2 + 1 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ -1 & -1 \end{bmatrix}$$

so the general solution to $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where $t$ is an arbitrary real number. Hence, the eigenvectors $\mathbf{x}$ corresponding to $\lambda_2$ are $\mathbf{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where $t \neq 0$ is arbitrary. Similarly, $\lambda_1 = 4$ gives rise to the eigenvectors $\mathbf{x} = t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $t \neq 0$ which includes the observation in Example 3.3.2.

Note that a square matrix $A$ has *many* eigenvectors associated with any given eigenvalue $\lambda$. In fact *every* nonzero solution $\mathbf{x}$ of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is an eigenvector. Recall that these solutions are all linear combinations of certain basic solutions determined by the gaussian algorithm (see Theorem 1.3.2). Observe that any nonzero multiple of an eigenvector is again an eigenvector,[9] and such multiples are often more convenient.[10] Any set of nonzero multiples of the basic solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ will be called a set of

[9] In fact, any nonzero linear combination of $\lambda$-eigenvectors is again a $\lambda$-eigenvector.

[10] Allowing nonzero multiples helps eliminate round-off error when the eigenvectors involve fractions.

**basic eigenvectors** corresponding to $\lambda$.

**Example 3.3.4**

Find the characteristic polynomial, eigenvalues, and basic eigenvectors for

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

**Solution.** Here the characteristic polynomial is given by

$$c_A(x) = \det \begin{bmatrix} x-2 & 0 & 0 \\ -1 & x-2 & 1 \\ -1 & -3 & x+2 \end{bmatrix} = (x-2)(x-1)(x+1)$$

so the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$. To find all eigenvectors for $\lambda_1 = 2$, compute

$$\lambda_1 I - A = \begin{bmatrix} \lambda_1 - 2 & 0 & 0 \\ -1 & \lambda_1 - 2 & 1 \\ -1 & -3 & \lambda_1 + 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -3 & 4 \end{bmatrix}$$

We want the (nonzero) solutions to $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$. The augmented matrix becomes

$$\left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

using row operations. Hence, the general solution $\mathbf{x}$ to $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ where $t$ is arbitrary, so we can use $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the basic eigenvector corresponding to $\lambda_1 = 2$. As the reader can verify, the gaussian algorithm gives basic eigenvectors $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 1 \end{bmatrix}$ corresponding to $\lambda_2 = 1$ and $\lambda_3 = -1$, respectively. Note that to eliminate fractions, we could instead use $3\mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ as the basic $\lambda_3$-eigenvector.

**Example 3.3.5**

If $A$ is a square matrix, show that $A$ and $A^T$ have the same characteristic polynomial, and hence the same eigenvalues.

**Solution.** We use the fact that $xI - A^T = (xI - A)^T$. Then

$$c_{A^T}(x) = \det\left(xI - A^T\right) = \det\left[(xI - A)^T\right] = \det(xI - A) = c_A(x)$$

by Theorem 3.2.3. Hence $c_{A^T}(x)$ and $c_A(x)$ have the same roots, and so $A^T$ and $A$ have the same eigenvalues (by Theorem 3.3.2).

The eigenvalues of a matrix need not be distinct. For example, if $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$ the characteristic polynomial is $(x-1)^2$ so the eigenvalue 1 occurs twice. Furthermore, eigenvalues are usually not computed as the roots of the characteristic polynomial. There are iterative, numerical methods (for example the QR-algorithm in Section 8.5) that are much more efficient for large matrices.

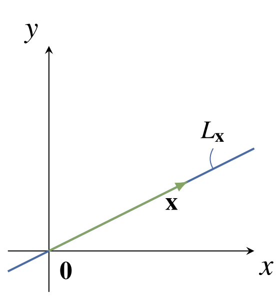
## $A$-Invariance

If $A$ is a $2 \times 2$ matrix, we can describe the eigenvectors of $A$ geometrically using the following concept. A line $L$ through the origin in $\mathbb{R}^2$ is called $A$-**invariant** if $A\mathbf{x}$ is in $L$ whenever $\mathbf{x}$ is in $L$. If we think of $A$ as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, this asks that $A$ carries $L$ into itself, that is the image $A\mathbf{x}$ of each vector $\mathbf{x}$ in $L$ is again in $L$.

**Example 3.3.6**

The $x$ axis $L = \left\{ \left[\begin{array}{c} x \\ 0 \end{array}\right] \mid x \text{ in } \mathbb{R} \right\}$ is $A$-invariant for any matrix of the form

$$A = \left[\begin{array}{cc} a & b \\ 0 & c \end{array}\right] \text{ because } \left[\begin{array}{cc} a & b \\ 0 & c \end{array}\right]\left[\begin{array}{c} x \\ 0 \end{array}\right] = \left[\begin{array}{c} ax \\ 0 \end{array}\right] \text{ is } L \text{ for all } \mathbf{x} = \left[\begin{array}{c} x \\ 0 \end{array}\right] \text{ in } L$$




To see the connection with eigenvectors, let $\mathbf{x} \neq \mathbf{0}$ be any nonzero vector in $\mathbb{R}^2$ and let $L_\mathbf{x}$ denote the unique line through the origin containing $\mathbf{x}$ (see the diagram). By the definition of scalar multiplication in Section 2.6, we see that $L_\mathbf{x}$ consists of all scalar multiples of $\mathbf{x}$, that is

$$L_\mathbf{x} = \mathbb{R}\mathbf{x} = \{t\mathbf{x} \mid t \text{ in } \mathbb{R}\}$$

Now suppose that $\mathbf{x}$ is an eigenvector of $A$, say $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda$ in $\mathbb{R}$. Then if $t\mathbf{x}$ is in $L_\mathbf{x}$ then

$$A(t\mathbf{x}) = t(A\mathbf{x}) = t(\lambda\mathbf{x}) = (t\lambda)\mathbf{x} \text{ is again in } L_\mathbf{x}$$

That is, $L_\mathbf{x}$ is $A$-invariant. On the other hand, if $L_\mathbf{x}$ is $A$-invariant then $A\mathbf{x}$ is in $L_\mathbf{x}$ (since $\mathbf{x}$ is in $L_\mathbf{x}$). Hence $A\mathbf{x} = t\mathbf{x}$ for some $t$ in $\mathbb{R}$, so $\mathbf{x}$ is an eigenvector for $A$ (with eigenvalue $t$). This proves:

**Theorem 3.3.3**

*Let $A$ be a $2 \times 2$ matrix, let $\mathbf{x} \neq \mathbf{0}$ be a vector in $\mathbb{R}^2$, and let $L_{\mathbf{x}}$ be the line through the origin in $\mathbb{R}^2$ containing $\mathbf{x}$. Then*

$$\mathbf{x} \text{ is an eigenvector of } A \quad \text{if and only if} \quad L_{\mathbf{x}} \text{ is } A\text{-invariant}$$

**Example 3.3.7**

1. If $\theta$ is not a multiple of $\pi$, show that $A = \left[ \begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right]$ has no real eigenvalue.

2. If $m$ is real show that $B = \frac{1}{1+m^2} \left[ \begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array} \right]$ has a 1 as an eigenvalue.

**Solution.**

1. $A$ induces rotation about the origin through the angle $\theta$ (Theorem 2.6.4). Since $\theta$ is not a multiple of $\pi$, this shows that no line through the origin is $A$-invariant. Hence $A$ has no eigenvector by Theorem 3.3.3, and so has no eigenvalue.

2. $B$ induces reflection $Q_m$ in the line through the origin with slope $m$ by Theorem 2.6.5. If $\mathbf{x}$ is any nonzero point on this line then it is clear that $Q_m\mathbf{x} = \mathbf{x}$, that is $Q_m\mathbf{x} = 1\mathbf{x}$. Hence 1 is an eigenvalue (with eigenvector $\mathbf{x}$).

If $\theta = \frac{\pi}{2}$ in Example 3.3.7, then $A = \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right]$ so $c_A(x) = x^2 + 1$. This polynomial has no root in $\mathbb{R}$, so $A$ has no (real) eigenvalue, and hence no eigenvector. In fact its eigenvalues are the complex numbers $i$ and $-i$, with corresponding eigenvectors $\left[ \begin{array}{c} 1 \\ -i \end{array} \right]$ and $\left[ \begin{array}{c} 1 \\ i \end{array} \right]$ In other words, $A$ *has* eigenvalues and eigenvectors, just not real ones.

Note that *every* polynomial has complex roots,[11] so every matrix has complex eigenvalues. While these eigenvalues may very well be real, this suggests that we really should be doing linear algebra over the complex numbers. Indeed, everything we have done (gaussian elimination, matrix algebra, determinants, etc.) works if all the scalars are complex.

[11] This is called the Fundamental Theorem of Algebra and was first proved by Gauss in his doctoral dissertation.

## Diagonalization

An $n \times n$ matrix $D$ is called a **diagonal matrix** if all its entries off the main diagonal are zero, that is if $D$ has the form

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \text{diag}\,(\lambda_1,\, \lambda_2,\, \cdots,\, \lambda_n)$$

where $\lambda_1,\, \lambda_2,\, \dots,\, \lambda_n$ are numbers. Calculations with diagonal matrices are very easy. Indeed, if $D = \text{diag}\,(\lambda_1,\, \lambda_2,\, \dots,\, \lambda_n)$ and $E = \text{diag}\,(\mu_1,\, \mu_2,\, \dots,\, \mu_n)$ are two diagonal matrices, their product $DE$ and sum $D+E$ are again diagonal, and are obtained by doing the same operations to corresponding diagonal elements:

$$\begin{aligned} DE &= \text{diag}\,(\lambda_1\mu_1,\, \lambda_2\mu_2,\, \dots,\, \lambda_n\mu_n) \\ D+E &= \text{diag}\,(\lambda_1+\mu_1,\, \lambda_2+\mu_2,\, \dots,\, \lambda_n+\mu_n) \end{aligned}$$

Because of the simplicity of these formulas, and with an eye on Theorem 3.3.1 and the discussion preceding it, we make another definition:

**Definition 3.6 Diagonalizable Matrices**

*An $n \times n$ matrix $A$ is called **diagonalizable** if*

$$P^{-1}AP \text{ is diagonal for some invertible } n \times n \text{ matrix } P$$

*Here the invertible matrix $P$ is called a **diagonalizing matrix** for $A$.*

To discover when such a matrix $P$ exists, we let $\mathbf{x}_1,\, \mathbf{x}_2,\, \dots,\, \mathbf{x}_n$ denote the columns of $P$ and look for ways to determine when such $\mathbf{x}_i$ exist and how to compute them. To this end, write $P$ in terms of its columns as follows:

$$P = [\mathbf{x}_1,\, \mathbf{x}_2,\, \cdots,\, \mathbf{x}_n]$$

Observe that $P^{-1}AP = D$ for some diagonal matrix $D$ holds if and only if

$$AP = PD$$

If we write $D = \text{diag}\,(\lambda_1,\, \lambda_2,\, \dots,\, \lambda_n)$, where the $\lambda_i$ are numbers to be determined, the equation $AP = PD$ becomes

$$A\,[\mathbf{x}_1,\, \mathbf{x}_2,\, \cdots,\, \mathbf{x}_n] = [\mathbf{x}_1,\, \mathbf{x}_2,\, \cdots,\, \mathbf{x}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

By the definition of matrix multiplication, each side simplifies as follows

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \end{bmatrix}$$

Comparing columns shows that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for each $i$, so

$$P^{-1}AP = D \quad \text{ if and only if } A\mathbf{x}_i = \lambda_i\mathbf{x}_i \text{ for each } i$$

In other words, $P^{-1}AP = D$ holds if and only if the diagonal entries of $D$ are eigenvalues of $A$ and the columns of $P$ are corresponding eigenvectors. This proves the following fundamental result.

**Theorem 3.3.4**

*Let $A$ be an $n \times n$ matrix.*

1. *$A$ is diagonalizable if and only if it has eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ such that the matrix $P = \left[ \begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{array} \right]$ is invertible.*
2. *When this is the case, $P^{-1}AP = \text{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$ where, for each $i$, $\lambda_i$ is the eigenvalue of $A$ corresponding to $\mathbf{x}_i$.*

**Example 3.3.8**

Diagonalize the matrix $A = \left[ \begin{array}{rrr} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{array} \right]$ in Example 3.3.4.

**Solution.** By Example 3.3.4, the eigenvalues of $A$ are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, with corresponding basic eigenvectors $\mathbf{x}_1 = \left[ \begin{array}{r} 1 \\ 1 \\ 1 \end{array} \right]$, $\mathbf{x}_2 = \left[ \begin{array}{r} 0 \\ 1 \\ 1 \end{array} \right]$, and $\mathbf{x}_3 = \left[ \begin{array}{r} 0 \\ 1 \\ 3 \end{array} \right]$ respectively. Since the matrix $P = \left[ \begin{array}{ccc} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{array} \right] = \left[ \begin{array}{rrr} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{array} \right]$ is invertible, Theorem 3.3.4 guarantees that

$$P^{-1}AP = \left[ \begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{array} \right] = \left[ \begin{array}{rrr} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] = D$$

The reader can verify this directly—easier to check $AP = PD$.

In Example 3.3.8, suppose we let $Q = \left[ \begin{array}{ccc} \mathbf{x}_2 & \mathbf{x}_1 & \mathbf{x}_3 \end{array} \right]$ be the matrix formed from the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ of $A$, but in a *different order* than that used to form $P$. Then $Q^{-1}AQ = \text{diag}\,(\lambda_2, \lambda_1, \lambda_3)$ is diagonal by Theorem 3.3.4, but the eigenvalues are in the *new* order. Hence we can choose the diagonalizing matrix $P$ so that the eigenvalues $\lambda_i$ appear in any order we want along the main diagonal of $D$.

In every example above each eigenvalue has had only one basic eigenvector. Here is a diagonalizable matrix where this is not the case.

**Example 3.3.9**

Diagonalize the matrix $A = \left[ \begin{array}{rrr} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \right]$

**Solution.** To compute the characteristic polynomial of $A$ first add rows 2 and 3 of $xI - A$ to row 1:

$$c_A(x) = \det \begin{bmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix} = \det \begin{bmatrix} x-2 & x-2 & x-2 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix}$$
$$= \det \begin{bmatrix} x-2 & 0 & 0 \\ -1 & x+1 & 0 \\ -1 & 0 & x+1 \end{bmatrix} = (x-2)(x+1)^2$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$, with $\lambda_2$ repeated twice (we say that $\lambda_2$ has *multiplicity* two). However, $A$ *is* diagonalizable. For $\lambda_1 = 2$, the system of equations $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ has general solution $\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the reader can verify, so a basic $\lambda_1$-eigenvector is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Turning to the repeated eigenvalue $\lambda_2 = -1$, we must solve $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$. By gaussian elimination, the general solution is $\mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ where $s$ and $t$ are arbitrary. Hence the gaussian algorithm produces *two* basic $\lambda_2$-eigenvectors $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ If we take $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ we find that $P$ is invertible. Hence $P^{-1}AP = \text{diag}\,(2,\ -1,\ -1)$ by Theorem 3.3.4.

Example 3.3.9 typifies every diagonalizable matrix. To describe the general case, we need some terminology.

**Definition 3.7 Multiplicity of an Eigenvalue**

*An eigenvalue $\lambda$ of a square matrix $A$ is said to have* ***multiplicity*** *$m$ if it occurs $m$ times as a root of the characteristic polynomial $c_A(x)$.*

For example, the eigenvalue $\lambda_2 = -1$ in Example 3.3.9 has multiplicity 2. In that example the gaussian algorithm yields two basic $\lambda_2$-eigenvectors, the same number as the multiplicity. This works in general.

**Theorem 3.3.5**

*A square matrix $A$ is diagonalizable if and only if every eigenvalue $\lambda$ of multiplicity $m$ yields exactly $m$ basic eigenvectors; that is, if and only if the general solution of the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has exactly $m$ parameters.*

One case of Theorem 3.3.5 deserves mention.

**Theorem 3.3.6**

*An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.*

The proofs of Theorem 3.3.5 and Theorem 3.3.6 require more advanced techniques and are given in Chapter 5. The following procedure summarizes the method.

**Diagonalization Algorithm**

*To diagonalize an $n \times n$ matrix $A$:*

*Step 1. Find the distinct eigenvalues $\lambda$ of $A$.*

*Step 2. Compute a set of basic eigenvectors corresponding to each of these eigenvalues $\lambda$ as basic solutions of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$.*

*Step 3. The matrix A is diagonalizable if and only if there are n basic eigenvectors in all.*

*Step 4. If $A$ is diagonalizable, the $n \times n$ matrix $P$ with these basic eigenvectors as its columns is a diagonalizing matrix for $A$, that is, $P$ is invertible and $P^{-1}AP$ is diagonal.*

The diagonalization algorithm is valid even if the eigenvalues are nonreal complex numbers. In this case the eigenvectors will also have complex entries, but we will not pursue this here.

**Example 3.3.10**

Show that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

**Solution 1.** The characteristic polynomial is $c_A(x) = (x-1)^2$, so $A$ has only one eigenvalue $\lambda_1 = 1$ of multiplicity 2. But the system of equations $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ has general solution $t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so there is only one parameter, and so only one basic eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Hence $A$ is not diagonalizable.

**Solution 2.** We have $c_A(x) = (x-1)^2$ so the only eigenvalue of $A$ is $\lambda = 1$. Hence, if $A$ were diagonalizable, Theorem 3.3.4 would give $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ for some invertible matrix $P$. But then $A = PIP^{-1} = I$, which is not the case. So $A$ cannot be diagonalizable.

Diagonalizable matrices share many properties of their eigenvalues. The following example illustrates why.

**Example 3.3.11**

If $\lambda^3 = 5\lambda$ for every eigenvalue of the diagonalizable matrix $A$, show that $A^3 = 5A$.

**Solution.** Let $P^{-1}AP = D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Because $\lambda_i^3 = 5\lambda_i$ for each $i$, we obtain

$$D^3 = \text{diag}(\lambda_1^3, \dots, \lambda_n^3) = \text{diag}(5\lambda_1, \dots, 5\lambda_n) = 5D$$

Hence $A^3 = (PDP^{-1})^3 = PD^3P^{-1} = P(5D)P^{-1} = 5(PDP^{-1}) = 5A$ using Theorem 3.3.1. This is what we wanted.

If $p(x)$ is any polynomial and $p(\lambda) = 0$ for every eigenvalue of the diagonalizable matrix $A$, an argument similar to that in Example 3.3.11 shows that $p(A) = 0$. Thus Example 3.3.11 deals with the case $p(x) = x^3 - 5x$. In general, $p(A)$ is called the *evaluation* of the polynomial $p(x)$ at the matrix $A$. For example, if $p(x) = 2x^3 - 3x + 5$, then $p(A) = 2A^3 - 3A + 5I$—note the use of the identity matrix.

In particular, if $c_A(x)$ denotes the characteristic polynomial of $A$, we certainly have $c_A(\lambda) = 0$ for each eigenvalue $\lambda$ of $A$ (Theorem 3.3.2). Hence $c_A(A) = 0$ for every diagonalizable matrix $A$. This is, in fact, true for *any* square matrix, diagonalizable or not, and the general result is called the Cayley-Hamilton theorem. It is proved in Section 8.7 and again in Section 11.1.

## Linear Dynamical Systems

We began Section 3.3 with an example from ecology which models the evolution of the population of a species of birds as time goes on. As promised, we now complete the example—Example 3.3.12 below.

The bird population was described by computing the female population profile $\mathbf{v}_k = \begin{bmatrix} a_k \\ j_k \end{bmatrix}$ of the species, where $a_k$ and $j_k$ represent the number of adult and juvenile females present $k$ years after the initial values $a_0$ and $j_0$ were observed. The model assumes that these numbers are related by the following equations:

$$\begin{aligned} a_{k+1} &= \tfrac{1}{2}a_k + \tfrac{1}{4}j_k \\ j_{k+1} &= 2a_k \end{aligned}$$

If we write $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$ the columns $\mathbf{v}_k$ satisfy $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for each $k = 0, 1, 2, \dots$.

Hence $\mathbf{v}_k = A^k\mathbf{v}_0$ for each $k = 1, 2, \dots$. We can now use our diagonalization techniques to determine the population profile $\mathbf{v}_k$ for all values of $k$ in terms of the initial values.

**Example 3.3.12**

Assuming that the initial values were $a_0 = 100$ adult females and $j_0 = 40$ juvenile females, compute $a_k$ and $j_k$ for $k = 1, 2, \dots$.

**Solution.** The characteristic polynomial of the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$ is

$c_A(x) = x^2 - \frac{1}{2}x - \frac{1}{2} = (x-1)(x+\frac{1}{2})$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$ and gaussian elimination gives corresponding basic eigenvectors $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$. For convenience, we can use multiples $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ respectively. Hence a diagonalizing matrix is $P = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ and we obtain

$$P^{-1}AP = D \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

This gives $A = PDP^{-1}$ so, for each $k \geq 0$, we can compute $A^k$ explicitly:

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^k \end{bmatrix} \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 4+2(-\frac{1}{2})^k & 1-(-\frac{1}{2})^k \\ 8-8(-\frac{1}{2})^k & 2+4(-\frac{1}{2})^k \end{bmatrix} \end{aligned}$$

Hence we obtain

$$\begin{aligned} \begin{bmatrix} a_k \\ j_k \end{bmatrix} = \mathbf{v}_k = A^k\mathbf{v}_0 &= \frac{1}{6} \begin{bmatrix} 4+2(-\frac{1}{2})^k & 1-(-\frac{1}{2})^k \\ 8-8(-\frac{1}{2})^k & 2+4(-\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} 100 \\ 40 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 440+160(-\frac{1}{2})^k \\ 880-640(-\frac{1}{2})^k \end{bmatrix} \end{aligned}$$

Equating top and bottom entries, we obtain exact formulas for $a_k$ and $j_k$:

$$a_k = \frac{220}{3} + \frac{80}{3}\left(-\frac{1}{2}\right)^k \text{ and } j_k = \frac{440}{3} + \frac{320}{3}\left(-\frac{1}{2}\right)^k \text{ for } k = 1,\ 2,\ \cdots$$

In practice, the exact values of $a_k$ and $j_k$ are not usually required. What is needed is a measure of how these numbers behave for large values of $k$. This is easy to obtain here. Since $(-\frac{1}{2})^k$ is nearly zero for large $k$, we have the following approximate values

$$a_k \approx \frac{220}{3} \text{ and } j_k \approx \frac{440}{3} \text{ if } k \text{ is large}$$

Hence, in the long term, the female population stabilizes with approximately twice as many juveniles as adults.

**Definition 3.8 Linear Dynamical System**

*If $A$ is an $n \times n$ matrix, a sequence $\mathbf{v}_0,\ \mathbf{v}_1,\ \mathbf{v}_2,\ \ldots$ of columns in $\mathbb{R}^n$ is called a **linear dynamical system** if $\mathbf{v}_0$ is specified and $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots$ are given by the matrix recurrence $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for each $k \geq 0$. We call $A$ the **migration** matrix of the system.*

We have $\mathbf{v}_1 = A\mathbf{v}_0$, then $\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$, and continuing we find

$$\mathbf{v}_k = A^k\mathbf{v}_0 \text{ for each } k = 1, 2, \cdots \tag{3.9}$$

Hence the columns $\mathbf{v}_k$ are determined by the powers $A^k$ of the matrix $A$ and, as we have seen, these powers can be efficiently computed if $A$ is diagonalizable. In fact Equation 3.9 can be used to give a nice "formula" for the columns $\mathbf{v}_k$ in this case.

Assume that $A$ is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding basic eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. If $P = \left[ \begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{array} \right]$ is a diagonalizing matrix with the $\mathbf{x}_i$ as columns, then $P$ is invertible and

$$P^{-1}AP = D = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

by Theorem 3.3.4. Hence $A = PDP^{-1}$ so Equation 3.9 and Theorem 3.3.1 give

$$\mathbf{v}_k = A^k\mathbf{v}_0 = (PDP^{-1})^k\mathbf{v}_0 = (PD^kP^{-1})\mathbf{v}_0 = PD^k(P^{-1}\mathbf{v}_0)$$

for each $k = 1, 2, \dots$. For convenience, we denote the column $P^{-1}\mathbf{v}_0$ arising here as follows:

$$\mathbf{b} = P^{-1}\mathbf{v}_0 = \left[ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]$$

Then matrix multiplication gives

$$\begin{aligned}
\mathbf{v}_k &= PD^k(P^{-1}\mathbf{v}_0) \\
&= \left[ \begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right] \left[ \begin{array}{cccc} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{array} \right] \left[ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right] \\
&= \left[ \begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right] \left[ \begin{array}{c} b_1\lambda_1^k \\ b_2\lambda_2^k \\ \vdots \\ b_3\lambda_n^k \end{array} \right] \\
&= b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n
\end{aligned} \tag{3.10}$$

for each $k \geq 0$. This is a useful **exact formula** for the columns $\mathbf{v}_k$. Note that, in particular,

$$\mathbf{v}_0 = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n$$

However, such an exact formula for $\mathbf{v}_k$ is often not required in practice; all that is needed is to *estimate* $\mathbf{v}_k$ for large values of $k$ (as was done in Example 3.3.12). This can be easily done if $A$ has a largest eigenvalue. An eigenvalue $\lambda$ of a matrix $A$ is called a **dominant eigenvalue** of $A$ if it has multiplicity 1 and

$$|\lambda| > |\mu| \text{ for all eigenvalues } \mu \neq \lambda$$

where $|\lambda|$ denotes the absolute value of the number $\lambda$. For example, $\lambda_1 = 1$ is dominant in Example 3.3.12.

Returning to the above discussion, suppose that $A$ has a dominant eigenvalue. By choosing the order in which the columns $\mathbf{x}_i$ are placed in $P$, we may assume that $\lambda_1$ is dominant among the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A$ (see the discussion following Example 3.3.8). Now recall the exact expression for $\mathbf{v}_k$ in Equation 3.10 above:

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n$$

Take $\lambda_1^k$ out as a common factor in this equation to get

$$\mathbf{v}_k = \lambda_1^k\left[b_1\mathbf{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{x}_n\right]$$

for each $k \geq 0$. Since $\lambda_1$ is dominant, we have $|\lambda_i| < |\lambda_1|$ for each $i \geq 2$, so each of the numbers $(\lambda_i/\lambda_1)^k$ become small in absolute value as $k$ increases. Hence $\mathbf{v}_k$ is approximately equal to the first term $\lambda_1^k b_1\mathbf{x}_1$, and we write this as $\mathbf{v}_k \approx \lambda_1^k b_1\mathbf{x}_1$. These observations are summarized in the following theorem (together with the above exact formula for $\mathbf{v}_k$).

**Theorem 3.3.7**

*Consider the dynamical system $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ with matrix recurrence*

$$\mathbf{v}_{k+1} = A\mathbf{v}_k \text{ for } k \geq 0$$

*where $A$ and $\mathbf{v}_0$ are given. Assume that $A$ is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding basic eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, and let $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{bmatrix}$ be the diagonalizing matrix. Then an exact formula for $\mathbf{v}_k$ is*

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \text{ for each } k \geq 0$$

*where the coefficients $b_i$ come from*

$$\mathbf{b} = P^{-1}\mathbf{v}_0 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

*Moreover, if $A$ has dominant[12] eigenvalue $\lambda_1$, then $\mathbf{v}_k$ is approximated by*

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 \text{ for sufficiently large } k.$$

**Example 3.3.13**

Returning to Example 3.3.12, we see that $\lambda_1 = 1$ is the dominant eigenvalue, with eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Here $P = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{v}_0 = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$ so $P^{-1}\mathbf{v}_0 = \frac{1}{3}\begin{bmatrix} 220 \\ -80 \end{bmatrix}$. Hence $b_1 = \frac{220}{3}$ in

[12] Similar results can be found in other situations. If for example, eigenvalues $\lambda_1$ and $\lambda_2$ (possibly equal) satisfy $|\lambda_1| = |\lambda_2| > |\lambda_i|$ for all $i > 2$, then we obtain $\mathbf{v}_k \approx b_1\lambda_1^k x_1 + b_2\lambda_2^k x_2$ for large $k$.

the notation of Theorem 3.3.7, so

$$\begin{bmatrix} a_k \\ j_k \end{bmatrix} = \mathbf{v}_k \approx b_1 \lambda_1^k \mathbf{x}_1 = \frac{220}{3} 1^k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where $k$ is large. Hence $a_k \approx \frac{220}{3}$ and $j_k \approx \frac{440}{3}$ as in Example 3.3.12.

This next example uses Theorem 3.3.7 to solve a "linear recurrence." See also Section 3.4.

**Example 3.3.14**

Suppose a sequence $x_0, x_1, x_2, \dots$ is determined by insisting that

$$x_0 = 1, \; x_1 = -1, \; \text{ and } x_{k+2} = 2x_k - x_{k+1} \text{ for every } k \geq 0$$

Find a formula for $x_k$ in terms of $k$.

**Solution.** Using the linear recurrence $x_{k+2} = 2x_k - x_{k+1}$ repeatedly gives

$$x_2 = 2x_0 - x_1 = 3, \quad x_3 = 2x_1 - x_2 = -5, \quad x_4 = 11, \quad x_5 = -21, \; \dots$$

so the $x_i$ are determined but no pattern is apparent. The idea is to find $\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$ for each $k$ instead, and then retrieve $x_k$ as the top component of $\mathbf{v}_k$. The reason this works is that the linear recurrence guarantees that these $\mathbf{v}_k$ are a dynamical system:

$$\mathbf{v}_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 2x_k - x_{k+1} \end{bmatrix} = A\mathbf{v}_k \text{ where } A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

The eigenvalues of $A$ are $\lambda_1 = -2$ and $\lambda_2 = 1$ with eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the diagonalizing matrix is $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$.

Moreover, $\mathbf{b} = P_0^{-1}\mathbf{v}_0 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ so the exact formula for $\mathbf{v}_k$ is

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \mathbf{v}_k = b_1 \lambda_1^k \mathbf{x}_1 + b_2 \lambda_2^k \mathbf{x}_2 = \frac{2}{3}(-2)^k \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{1}{3} 1^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Equating top entries gives the desired formula for $x_k$:

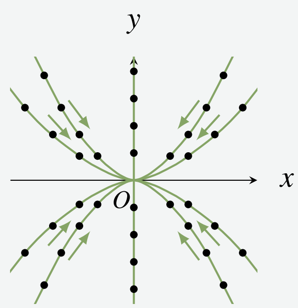
$$x_k = \frac{1}{3}\left[2(-2)^k + 1\right] \text{ for all } k = 0, \; 1, \; 2, \; \dots$$

The reader should check this for the first few values of $k$.

## Graphical Description of Dynamical Systems

If a dynamical system $\mathbf{v}_{k+1} = A\mathbf{v}_k$ is given, the sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ is called the **trajectory** of the system starting at $\mathbf{v}_0$. It is instructive to obtain a graphical plot of the system by writing $\mathbf{v}_k = \left[ \begin{array}{c} x_k \\ y_k \end{array} \right]$ and plotting the successive values as points in the plane, identifying $\mathbf{v}_k$ with the point $(x_k, y_k)$ in the plane. We give several examples which illustrate properties of dynamical systems. For ease of calculation we assume that the matrix $A$ is simple, usually diagonal.

**Example 3.3.15**

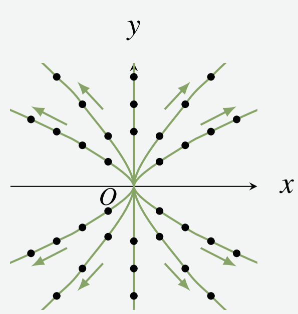



Let $A = \left[ \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{array} \right]$ Then the eigenvalues are $\frac{1}{2}$ and $\frac{1}{3}$, with corresponding eigenvectors $\mathbf{x}_1 = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$ and $\mathbf{x}_2 = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$. The exact formula is

$$\mathbf{v}_k = b_1 \left(\frac{1}{2}\right)^k \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + b_2 \left(\frac{1}{3}\right)^k \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$$

for $k = 0, 1, 2, \dots$ by Theorem 3.3.7, where the coefficients $b_1$ and $b_2$ depend on the initial point $\mathbf{v}_0$. Several trajectories are plotted in the diagram and, for each choice of $\mathbf{v}_0$, the trajectories converge toward the origin because both eigenvalues are less than 1 in absolute value. For this reason, the origin is called an **attractor** for the system.

**Example 3.3.16**

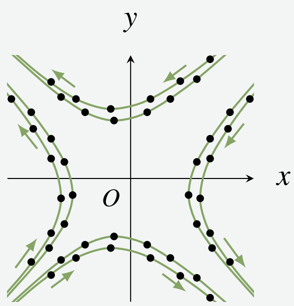



Let $A = \left[ \begin{array}{cc} \frac{3}{2} & 0 \\ 0 & \frac{4}{3} \end{array} \right]$. Here the eigenvalues are $\frac{3}{2}$ and $\frac{4}{3}$, with corresponding eigenvectors $\mathbf{x}_1 = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$ and $\mathbf{x}_2 = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$ as before. The exact formula is

$$\mathbf{v}_k = b_1 \left(\frac{3}{2}\right)^k \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + b_2 \left(\frac{4}{3}\right)^k \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$$

for $k = 0, 1, 2, \dots$. Since both eigenvalues are greater than 1 in absolute value, the trajectories diverge away from the origin for every choice of initial point $V_0$. For this reason, the origin is called a **repellor** for the system.

**Example 3.3.17**




Let $A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$. Now the eigenvalues are $\frac{3}{2}$ and $\frac{1}{2}$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ The exact formula is

$$\mathbf{v}_k = b_1 \left(\frac{3}{2}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b_2 \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

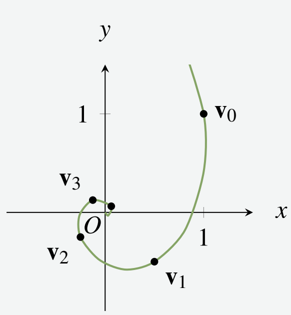
for $k = 0,\ 1,\ 2,\ \dots$. In this case $\frac{3}{2}$ is the dominant eigenvalue so, if $b_1 \neq 0$, we have $\mathbf{v}_k \approx b_1 \left(\frac{3}{2}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for large $k$ and $\mathbf{v}_k$ is approaching the line $y = -x$.

However, if $b_1 = 0$, then $\mathbf{v}_k = b_2 \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and so approaches the origin along the line $y = x$. In general the trajectories appear as in the diagram, and the origin is called a **saddle point** for the dynamical system in this case.

**Example 3.3.18**

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$. Now the characteristic polynomial is $c_A(x) = x^2 + \frac{1}{4}$, so the eigenvalues are the complex numbers $\frac{i}{2}$ and $-\frac{i}{2}$ where $i^2 = -1$. Hence $A$ is not diagonalizable as a real matrix. However, the trajectories are not difficult to describe. If we start with $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then the trajectory begins as

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{8} \end{bmatrix},\ \mathbf{v}_4 = \begin{bmatrix} \frac{1}{16} \\ \frac{1}{16} \end{bmatrix},\ \mathbf{v}_5 = \begin{bmatrix} \frac{1}{32} \\ -\frac{1}{32} \end{bmatrix},\ \mathbf{v}_6 = \begin{bmatrix} -\frac{1}{64} \\ -\frac{1}{64} \end{bmatrix},\ \dots$$




The first five of these points are plotted in the diagram. Here each trajectory spirals in toward the origin, so the origin is an attractor. Note that the two (complex) eigenvalues have absolute value less than 1 here. If they had absolute value greater than 1, the trajectories would spiral out from the origin.

## Google PageRank

Dominant eigenvalues are useful to the Google search engine for finding information on the Web. If an information query comes in from a client, Google has a sophisticated method of establishing the "relevance" of each site to that query. When the relevant sites have been determined, they are placed in order of importance using a ranking of *all* sites called the PageRank. The relevant sites with the highest PageRank are the ones presented to the client. It is the construction of the PageRank that is our interest here.

The Web contains many links from one site to another. Google interprets a link from site $j$ to site $i$ as a "vote" for the importance of site $i$. Hence if site $i$ has more links to it than does site $j$, then $i$ is regarded as more "important" and assigned a higher PageRank. One way to look at this is to view the sites as vertices in a huge directed graph (see Section 2.2). Then if site $j$ links to site $i$ there is an edge from $j$ to $i$, and hence the $(i, j)$-entry is a 1 in the associated adjacency matrix (called the *connectivity* matrix in this context). Thus a large number of 1s in row $i$ of this matrix is a measure of the PageRank of site $i$.[13]

However this does not take into account the PageRank of the sites that link to $i$. Intuitively, the higher the rank of these sites, the higher the rank of site $i$. One approach is to compute a dominant eigenvector $\mathbf{x}$ for the connectivity matrix. In most cases the entries of $\mathbf{x}$ can be chosen to be positive with sum 1. Each site corresponds to an entry of $\mathbf{x}$, so the sum of the entries of sites linking to a given site $i$ is a measure of the rank of site $i$. In fact, Google chooses the PageRank of a site so that it is proportional to this sum.[14]

# Exercises for 3.3

**Exercise 3.3.1** In each case find the characteristic polynomial, eigenvalues, eigenvectors, and (if possible) an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.

a. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

c. $A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 5 \end{bmatrix}$

e. $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$

f. $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

g. $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$

h. $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

i. $A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \lambda \neq \mu$

**Exercise 3.3.2** Consider a linear dynamical system $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for $k \geq 0$. In each case approximate $\mathbf{v}_k$ using Theorem 3.3.7.

a. $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

**Exercise 3.3.3** Show that $A$ has $\lambda = 0$ as an eigenvalue if and only if $A$ is not invertible.

**Exercise 3.3.4** Let $A$ denote an $n \times n$ matrix and put $A_1 = A - \alpha I$, $\alpha$ in $\mathbb{R}$. Show that $\lambda$ is an eigenvalue of $A$ if and only if $\lambda - \alpha$ is an eigenvalue of $A_1$. (Hence,

[13] For more on PageRank, visit https://en.wikipedia.org/wiki/PageRank.

[14] See the articles "Searching the web with eigenvectors" by Herbert S. Wilf, UMAP Journal 23(2), 2002, pages 101–103, and "The worlds largest matrix computation: Google's PageRank is an eigenvector of a matrix of order 2.7 billion" by Cleve Moler, Matlab News and Notes, October 2002, pages 12–13.

the eigenvalues of $A_1$ are just those of $A$ "shifted" by $\alpha$.) How do the eigenvectors compare?

**Exercise 3.3.5** Show that the eigenvalues of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $e^{i\theta}$ and $e^{-i\theta}$. (See Appendix A)

**Exercise 3.3.6** Find the characteristic polynomial of the $n \times n$ identity matrix $I$. Show that $I$ has exactly one eigenvalue and find the eigenvectors.

**Exercise 3.3.7** Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that:

a. $c_A(x) = x^2 - \text{tr}\, Ax + \det A$, where $\text{tr}\, A = a + d$ is called the **trace** of $A$.

b. The eigenvalues are $\frac{1}{2}\left[(a+d) \pm \sqrt{(a-b)^2 + 4bc}\right]$.

**Exercise 3.3.8** In each case, find $P^{-1}AP$ and then compute $A^n$.

a. $A = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} -7 & -12 \\ 6 & -10 \end{bmatrix}$, $P = \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}$

[*Hint*: $(PDP^{-1})^n = PD^nP^{-1}$ for each $n = 1, 2, \dots$.]

**Exercise 3.3.9**

a. If $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ verify that $A$ and $B$ are diagonalizable, but $AB$ is not.

b. If $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ find a diagonalizable matrix $A$ such that $D + A$ is not diagonalizable.

**Exercise 3.3.10** If $A$ is an $n \times n$ matrix, show that $A$ is diagonalizable if and only if $A^T$ is diagonalizable.

**Exercise 3.3.11** If $A$ is diagonalizable, show that each of the following is also diagonalizable.

a. $A^n$, $n \geq 1$

b. $kA$, $k$ any scalar.

c. $p(A)$, $p(x)$ any polynomial (Theorem 3.3.1)

d. $U^{-1}AU$ for any invertible matrix $U$.

e. $kI + A$ for any scalar $k$.

**Exercise 3.3.12** Give an example of two diagonalizable matrices $A$ and $B$ whose sum $A + B$ is not diagonalizable.

**Exercise 3.3.13** If $A$ is diagonalizable and 1 and $-1$ are the only eigenvalues, show that $A^{-1} = A$.

**Exercise 3.3.14** If $A$ is diagonalizable and 0 and 1 are the only eigenvalues, show that $A^2 = A$.

**Exercise 3.3.15** If $A$ is diagonalizable and $\lambda \geq 0$ for each eigenvalue of $A$, show that $A = B^2$ for some matrix $B$.

**Exercise 3.3.16** If $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, show that $AB = BA$. [*Hint*: Diagonal matrices commute.]

**Exercise 3.3.17** A square matrix $A$ is called **nilpotent** if $A^n = 0$ for some $n \geq 1$. Find all nilpotent diagonalizable matrices. [*Hint*: Theorem 3.3.1.]

**Exercise 3.3.18** Let $A$ be any $n \times n$ matrix and $r \neq 0$ a real number.

a. Show that the eigenvalues of $rA$ are precisely the numbers $r\lambda$, where $\lambda$ is an eigenvalue of $A$.

b. Show that $c_{rA}(x) = r^n c_A\left(\frac{x}{r}\right)$.

**Exercise 3.3.19**

a. If all rows of $A$ have the same sum $s$, show that $s$ is an eigenvalue.

b. If all columns of $A$ have the same sum $s$, show that $s$ is an eigenvalue.

**Exercise 3.3.20** Let $A$ be an invertible $n \times n$ matrix.

a. Show that the eigenvalues of $A$ are nonzero.

b. Show that the eigenvalues of $A^{-1}$ are precisely the numbers $1/\lambda$, where $\lambda$ is an eigenvalue of $A$.

c. Show that $c_{A^{-1}}(x) = \frac{(-x)^n}{\det A} c_A\left(\frac{1}{x}\right)$.

**Exercise 3.3.21** Suppose $\lambda$ is an eigenvalue of a square matrix $A$ with eigenvector $\mathbf{x} \neq \mathbf{0}$.

a. Show that $\lambda^2$ is an eigenvalue of $A^2$ (with the same $\mathbf{x}$).

b. Show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$.

c. Show that $p(\lambda)$ is an eigenvalue of $p(A)$ for any nonzero polynomial $p(x)$.

**Exercise 3.3.22** If $A$ is an $n \times n$ matrix, show that $c_{A^2}(x^2) = (-1)^n c_A(x) c_A(-x)$.

**Exercise 3.3.23** An $n \times n$ matrix $A$ is called nilpotent if $A^m = 0$ for some $m \geq 1$.

a. Show that every triangular matrix with zeros on the main diagonal is nilpotent.

b. If $A$ is nilpotent, show that $\lambda = 0$ is the only eigenvalue (even complex) of $A$.

c. Deduce that $c_A(x) = x^n$, if $A$ is $n \times n$ and nilpotent.

**Exercise 3.3.24** Let $A$ be diagonalizable with real eigenvalues and assume that $A^m = I$ for some $m \geq 1$.

a. Show that $A^2 = I$.

b. If $m$ is odd, show that $A = I$.

[*Hint*: Theorem A.3]

**Exercise 3.3.25** Let $A^2 = I$, and assume that $A \neq I$ and $A \neq -I$.

a. Show that the only eigenvalues of $A$ are $\lambda = 1$ and $\lambda = -1$.

b. Show that $A$ is diagonalizable. [*Hint*: Verify that $A(A+I) = A+I$ and $A(A-I) = -(A-I)$, and then look at nonzero columns of $A+I$ and of $A-I$.]

c. If $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = mx$ where $m \neq 0$, use (b) to show that the matrix of $Q_m$ is diagonalizable for each $m$.

d. Now prove (c) geometrically using Theorem 3.3.3.

**Exercise 3.3.26** Let $A = \left[ \begin{array}{rrr} 2 & 3 & -3 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{array} \right]$ and $B = \left[ \begin{array}{rrr} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{array} \right]$. Show that $c_A(x) = c_B(x) = (x+1)^2(x-2)$, but $A$ is diagonalizable and $B$ is not.

**Exercise 3.3.27**

a. Show that the only diagonalizable matrix $A$ that has only one eigenvalue $\lambda$ is the scalar matrix $A = \lambda I$.

b. Is $\left[ \begin{array}{rr} 3 & -2 \\ 2 & -1 \end{array} \right]$ diagonalizable?

**Exercise 3.3.28** Characterize the diagonalizable $n \times n$ matrices $A$ such that $A^2 - 3A + 2I = 0$ in terms of their eigenvalues. [*Hint*: Theorem 3.3.1.]

**Exercise 3.3.29** Let $A = \left[ \begin{array}{cc} B & 0 \\ 0 & C \end{array} \right]$ where $B$ and $C$ are square matrices.

a. If $B$ and $C$ are diagonalizable via $Q$ and $R$ (that is, $Q^{-1}BQ$ and $R^{-1}CR$ are diagonal), show that $A$ is diagonalizable via $\left[ \begin{array}{cc} Q & 0 \\ 0 & R \end{array} \right]$

b. Use (a) to diagonalize $A$ if $B = \left[ \begin{array}{rr} 5 & 3 \\ 3 & 5 \end{array} \right]$ and $C = \left[ \begin{array}{rr} 7 & -1 \\ -1 & 7 \end{array} \right]$.

**Exercise 3.3.30** Let $A = \left[ \begin{array}{cc} B & 0 \\ 0 & C \end{array} \right]$ where $B$ and $C$ are square matrices.

a. Show that $c_A(x) = c_B(x)c_C(x)$.

b. If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of $B$ and $C$, respectively, show that $\left[\begin{array}{c}\mathbf{x}\\0\end{array}\right]$ and $\left[\begin{array}{c}0\\\mathbf{y}\end{array}\right]$ are eigenvectors of $A$, and show how every eigenvector of $A$ arises from such eigenvectors.

**Exercise 3.3.31** Referring to the model in Example 3.3.1, determine if the population stabilizes, becomes extinct, or becomes large in each case. Denote the adult and juvenile survival rates as $A$ and $J$, and the reproduction rate as $R$.

| | $R$ | $A$ | $J$ |
|---|---|---|---|
| $a.$ | 2 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $b.$ | 3 | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $c.$ | 2 | $\frac{1}{4}$ | $\frac{1}{3}$ |
| $d.$ | 3 | $\frac{3}{5}$ | $\frac{1}{5}$ |

**Exercise 3.3.32** In the model of Example 3.3.1, does the final outcome depend on the initial population of adult and juvenile females? Support your answer.

**Exercise 3.3.33** In Example 3.3.1, keep the same reproduction rate of 2 and the same adult survival rate of $\frac{1}{2}$, but suppose that the juvenile survival rate is $\rho$. Determine which values of $\rho$ cause the population to become extinct or to become large.

**Exercise 3.3.34** In Example 3.3.1, let the juvenile survival rate be $\frac{2}{5}$ and let the reproduction rate be 2. What values of the adult survival rate $\alpha$ will ensure that the population stabilizes?

## 3.4 An Application to Linear Recurrences

It often happens that a problem can be solved by finding a sequence of numbers $x_0, x_1, x_2, \dots$ where the first few are known, and subsequent numbers are given in terms of earlier ones. Here is a combinatorial example where the object is to count the number of ways to do something.

**Example 3.4.1**

An urban planner wants to determine the number $x_k$ of ways that a row of $k$ parking spaces can be filled with cars and trucks if trucks take up two spaces each. Find the first few values of $x_k$.

**Solution.** Clearly, $x_0 = 1$ and $x_1 = 1$, while $x_2 = 2$ since there can be two cars or one truck. We have $x_3 = 3$ (the 3 configurations are *ccc*, *cT*, and *Tc*) and $x_4 = 5$ (*cccc, ccT, cTc, Tcc,* and *TT*). The key to this method is to find a way to express each subsequent $x_k$ in terms of earlier values. In this case we claim that

$$x_{k+2} = x_k + x_{k+1} \text{ for every } k \geq 0 \tag{3.11}$$

Indeed, every way to fill $k+2$ spaces falls into one of two categories: Either a car is parked in the first space (and the remaining $k+1$ spaces are filled in $x_{k+1}$ ways), or a truck is parked in the first two spaces (with the other $k$ spaces filled in $x_k$ ways). Hence, there are $x_{k+1} + x_k$ ways to fill the $k+2$ spaces. This is Equation 3.11.

The recurrence in Equation 3.11 determines $x_k$ for every $k \geq 2$ since $x_0$ and $x_1$ are given. In fact, the first few values are

$$\begin{array}{rclcl} x_0 & = & 1 & & \\ x_1 & = & 1 & & \\ x_2 & = & x_0 + x_1 & = & 2 \\ x_3 & = & x_1 + x_2 & = & 3 \\ x_4 & = & x_2 + x_3 & = & 5 \\ x_5 & = & x_3 + x_4 & = & 8 \\ \vdots & & \vdots & & \vdots \end{array}$$

Clearly, we can find $x_k$ for any value of $k$, but one wishes for a "formula" for $x_k$ as a function of $k$. It turns out that such a formula can be found using diagonalization. We will return to this example later.

A sequence $x_0, x_1, x_2, \dots$ of numbers is said to be given **recursively** if each number in the sequence is completely determined by those that come before it. Such sequences arise frequently in mathematics and computer science, and also occur in other parts of science. The formula $x_{k+2} = x_{k+1} + x_k$ in Example 3.4.1 is an example of a **linear recurrence relation** of length 2 because $x_{k+2}$ is the sum of the two preceding terms $x_{k+1}$ and $x_k$; in general, the **length** is $m$ if $x_{k+m}$ is a sum of multiples of $x_k, x_{k+1}, \dots, x_{k+m-1}$.

The simplest linear recursive sequences are of length 1, that is $x_{k+1}$ is a fixed multiple of $x_k$ for each $k$, say $x_{k+1} = ax_k$. If $x_0$ is specified, then $x_1 = ax_0$, $x_2 = ax_1 = a^2x_0$, and $x_3 = ax_2 = a^3x_0, \dots$. Continuing, we obtain $x_k = a^kx_0$ for each $k \geq 0$, which is an explicit formula for $x_k$ as a function of $k$ (when $x_0$ is given).

Such formulas are not always so easy to find for all choices of the initial values. Here is an example where diagonalization helps.

**Example 3.4.2**

Suppose the numbers $x_0, x_1, x_2, \dots$ are given by the linear recurrence relation

$$x_{k+2} = x_{k+1} + 6x_k \text{ for } k \geq 0$$

where $x_0$ and $x_1$ are specified. Find a formula for $x_k$ when $x_0 = 1$ and $x_1 = 3$, and also when $x_0 = 1$ and $x_1 = 1$.

**Solution.** If $x_0 = 1$ and $x_1 = 3$, then

$$x_2 = x_1 + 6x_0 = 9, \quad x_3 = x_2 + 6x_1 = 27, \quad x_4 = x_3 + 6x_2 = 81$$

and it is apparent that

$$x_k = 3^k \text{ for } k = 0,\ 1,\ 2,\ 3, \text{ and } 4$$

This formula holds for all $k$ because it is true for $k = 0$ and $k = 1$, and it satisfies the recurrence $x_{k+2} = x_{k+1} + 6x_k$ for each $k$ as is readily checked.

However, if we begin instead with $x_0 = 1$ and $x_1 = 1$, the sequence continues

$$x_2 = 7, \quad x_3 = 13, \quad x_4 = 55, \quad x_5 = 133, \quad \dots$$

In this case, the sequence is uniquely determined but no formula is apparent. Nonetheless, a simple device transforms the recurrence into a matrix recurrence to which our diagonalization techniques apply.

The idea is to compute the sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ of columns instead of the numbers $x_0, x_1, x_2, \dots,$ where

$$\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} \text{ for each } k \geq 0$$

Then $\mathbf{v}_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is specified, and the numerical recurrence $x_{k+2} = x_{k+1} + 6x_k$ transforms into a matrix recurrence as follows:

$$\mathbf{v}_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 6x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A\mathbf{v}_k$$

where $A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$. Thus these columns $\mathbf{v}_k$ are a linear dynamical system, so Theorem 3.3.7 applies provided the matrix $A$ is diagonalizable.
We have $c_A(x) = (x-3)(x+2)$ so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as the reader can check. Since $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$ is invertible, it is a diagonalizing matrix for $A$. The coefficients $b_i$ in Theorem 3.3.7 are given by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}\mathbf{v}_0 = \begin{bmatrix} \frac{3}{5} \\ \frac{-2}{5} \end{bmatrix}$, so that the theorem gives

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 = \tfrac{3}{5}3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \tfrac{-2}{5}(-2)^k \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Equating top entries yields

$$x_k = \tfrac{1}{5}\left[3^{k+1} - (-2)^{k+1}\right] \text{ for } k \geq 0$$

This gives $x_0 = 1 = x_1$, and it satisfies the recurrence $x_{k+2} = x_{k+1} + 6x_k$ as is easily verified. Hence, it is the desired formula for the $x_k$.

Returning to Example 3.4.1, these methods give an exact formula and a good approximation for the numbers $x_k$ in that problem.

**Example 3.4.3**

In Example 3.4.1, an urban planner wants to determine $x_k$, the number of ways that a row of $k$ parking spaces can be filled with cars and trucks if trucks take up two spaces each. Find a formula for $x_k$ and estimate it for large $k$.

**Solution.** We saw in Example 3.4.1 that the numbers $x_k$ satisfy a linear recurrence

$$x_{k+2} = x_k + x_{k+1} \text{ for every } k \geq 0$$

If we write $\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$ as before, this recurrence becomes a matrix recurrence for the $\mathbf{v}_k$:

$$\mathbf{v}_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A\mathbf{v}_k$$

for all $k \geq 0$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Moreover, $A$ is diagonalizable here. The characteristic polynomial is $c_A(x) = x^2 - x - 1$ with roots $\frac{1}{2}\left[1 \pm \sqrt{5}\right]$ by the quadratic formula, so $A$ has eigenvalues

$$\lambda_1 = \tfrac{1}{2}\left(1 + \sqrt{5}\right) \text{ and } \lambda_2 = \tfrac{1}{2}\left(1 - \sqrt{5}\right)$$

Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$ respectively as the reader can verify.
As the matrix $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$ is invertible, it is a diagonalizing matrix for $A$. We compute the coefficients $b_1$ and $b_2$ (in Theorem 3.3.7) as follows:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}\mathbf{v}_0 = \frac{1}{-\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 \\ -\lambda_2 \end{bmatrix}$$

where we used the fact that $\lambda_1 + \lambda_2 = 1$. Thus Theorem 3.3.7 gives

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 = \frac{\lambda_1}{\sqrt{5}}\lambda_1^k \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} - \frac{\lambda_2}{\sqrt{5}}\lambda_2^k \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

Comparing top entries gives an exact formula for the numbers $x_k$:

$$x_k = \frac{1}{\sqrt{5}}\left[\lambda_1^{k+1} - \lambda_2^{k+1}\right] \text{ for } k \geq 0$$

Finally, observe that $\lambda_1$ is dominant here (in fact, $\lambda_1 = 1.618$ and $\lambda_2 = -0.618$ to three decimal places) so $\lambda_2^{k+1}$ is negligible compared with $\lambda_1^{k+1}$ is large. Thus,

$$x_k \approx \frac{1}{\sqrt{5}}\lambda_1^{k+1} \text{ for each } k \geq 0.$$

This is a good approximation, even for as small a value as $k = 12$. Indeed, repeated use of the recurrence $x_{k+2} = x_k + x_{k+1}$ gives the exact value $x_{12} = 233$, while the approximation is $x_{12} \approx \frac{(1.618)^{13}}{\sqrt{5}} = 232.94$.

The sequence $x_0, x_1, x_2, \dots$ in Example 3.4.3 was first discussed in 1202 by Leonardo Pisano of Pisa, also known as Fibonacci,[15] and is now called the **Fibonacci sequence**. It is completely determined by the conditions $x_0 = 1$, $x_1 = 1$ and the recurrence $x_{k+2} = x_k + x_{k+1}$ for each $k \geq 0$. These numbers have

[15]Fibonacci was born in Italy. As a young man he travelled to India where he encountered the “Fibonacci” sequence. He returned to Italy and published this in his book Liber Abaci in 1202. In the book he is the first to bring the Hindu decimal system for representing numbers to Europe.

been studied for centuries and have many interesting properties (there is even a journal, the *Fibonacci Quarterly*, devoted exclusively to them). For example, biologists have discovered that the arrangement of leaves around the stems of some plants follow a Fibonacci pattern. The formula $x_k = \frac{1}{\sqrt{5}}\left[\lambda_1^{k+1} - \lambda_2^{k+1}\right]$ in Example 3.4.3 is called the **Binet formula**. It is remarkable in that the $x_k$ are integers but $\lambda_1$ and $\lambda_2$ are not. This phenomenon can occur even if the eigenvalues $\lambda_i$ are nonreal complex numbers.

We conclude with an example showing that *nonlinear* recurrences can be very complicated.

**Example 3.4.4**

Suppose a sequence $x_0, x_1, x_2, \dots$ satisfies the following recurrence:

$$x_{k+1} = \begin{cases} \frac{1}{2}x_k & \text{if } x_k \text{ is even} \\ 3x_k + 1 & \text{if } x_k \text{ is odd} \end{cases}$$

If $x_0 = 1$, the sequence is $1, 4, 2, 1, 4, 2, 1, \dots$ and so continues to cycle indefinitely. The same thing happens if $x_0 = 7$. Then the sequence is

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \dots$$

and it again cycles. However, it is not known whether every choice of $x_0$ will lead eventually to 1. It is quite possible that, for some $x_0$, the sequence will continue to produce different values indefinitely, or will repeat a value and cycle without reaching 1. No one knows for sure.

## Exercises for 3.4

**Exercise 3.4.1** Solve the following linear recurrences.

a. $x_{k+2} = 3x_k + 2x_{k+1}$, where $x_0 = 1$ and $x_1 = 1$.

b. $x_{k+2} = 2x_k - x_{k+1}$, where $x_0 = 1$ and $x_1 = 2$.

c. $x_{k+2} = 2x_k + x_{k+1}$, where $x_0 = 0$ and $x_1 = 1$.

d. $x_{k+2} = 6x_k - x_{k+1}$, where $x_0 = 1$ and $x_1 = 1$.

**Exercise 3.4.2** Solve the following linear recurrences.

a. $x_{k+3} = 6x_{k+2} - 11x_{k+1} + 6x_k$, where $x_0 = 1$, $x_1 = 0$, and $x_2 = 1$.

b. $x_{k+3} = -2x_{k+2} + x_{k+1} + 2x_k$, where $x_0 = 1$, $x_1 = 0$, and $x_2 = 1$.

[*Hint*: Use $\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix}$.]

**Exercise 3.4.3** In Example 3.4.1 suppose buses are also allowed to park, and let $x_k$ denote the number of ways a row of $k$ parking spaces can be filled with cars, trucks, and buses.

a. If trucks and buses take up 2 and 3 spaces respectively, show that $x_{k+3} = x_k + x_{k+1} + x_{k+2}$ for each $k$, and use this recurrence to compute $x_{10}$. [*Hint*: The eigenvalues are of little use.]

b. If buses take up 4 spaces, find a recurrence for the $x_k$ and compute $x_{10}$.

**Exercise 3.4.4** A man must climb a flight of $k$ steps. He always takes one or two steps at a time. Thus he can climb 3 steps in the following ways: 1, 1, 1; 1, 2; or 2, 1. Find $s_k$, the number of ways he can climb the flight of $k$ steps. [*Hint*: Fibonacci.]

**Exercise 3.4.5** How many "words" of $k$ letters can be made from the letters $\{a, b\}$ if there are no adjacent $a$'s?

**Exercise 3.4.6** How many sequences of $k$ flips of a coin are there with no $HH$?

**Exercise 3.4.7** Find $x_k$, the number of ways to make a stack of $k$ poker chips if only red, blue, and gold chips are used and no two gold chips are adjacent. [*Hint*: Show that $x_{k+2} = 2x_{k+1} + 2x_k$ by considering how many stacks have a red, blue, or gold chip on top.]

**Exercise 3.4.8** A nuclear reactor contains $\alpha$- and $\beta$-particles. In every second each $\alpha$-particle splits into three $\beta$-particles, and each $\beta$-particle splits into an $\alpha$-particle and two $\beta$-particles. If there is a single $\alpha$-particle in the reactor at time $t = 0$, how many $\alpha$-particles are there at $t = 20$ seconds? [*Hint*: Let $x_k$ and $y_k$ denote the number of $\alpha$- and $\beta$-particles at time $t = k$ seconds. Find $x_{k+1}$ and $y_{k+1}$ in terms of $x_k$ and $y_k$.]

**Exercise 3.4.9** The annual yield of wheat in a certain country has been found to equal the average of the yield in the previous two years. If the yields in 1990 and 1991 were 10 and 12 million tons respectively, find a formula for the yield $k$ years after 1990. What is the long-term average yield?

**Exercise 3.4.10** Find the general solution to the recurrence $x_{k+1} = rx_k + c$ where $r$ and $c$ are constants. [*Hint*: Consider the cases $r = 1$ and $r \neq 1$ separately. If $r \neq 1$, you will need the identity $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$ for $n \geq 1$.]

**Exercise 3.4.11** Consider the length 3 recurrence $x_{k+3} = ax_k + bx_{k+1} + cx_{k+2}$.

a. If $\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ show that $\mathbf{v}_{k+1} = A\mathbf{v}_k$.

b. If $\lambda$ is any eigenvalue of $A$, show that $\mathbf{x} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$ is a $\lambda$-eigenvector.

[*Hint*: Show directly that $A\mathbf{x} = \lambda\mathbf{x}$.]

c. Generalize (a) and (b) to a recurrence

$$x_{k+4} = ax_k + bx_{k+1} + cx_{k+2} + dx_{k+3}$$

of length 4.

**Exercise 3.4.12** Consider the recurrence

$$x_{k+2} = ax_{k+1} + bx_k + c$$

where $c$ may not be zero.

a. If $a + b \neq 1$ show that $p$ can be found such that, if we set $y_k = x_k + p$, then $y_{k+2} = ay_{k+1} + by_k$. [Hence, the sequence $x_k$ can be found provided $y_k$ can be found by the methods of this section (or otherwise).]

b. Use (a) to solve $x_{k+2} = x_{k+1} + 6x_k + 5$ where $x_0 = 1$ and $x_1 = 1$.

**Exercise 3.4.13** Consider the recurrence

$$x_{k+2} = ax_{k+1} + bx_k + c(k) \tag{3.12}$$

where $c(k)$ is a function of $k$, and consider the related recurrence

$$x_{k+2} = ax_{k+1} + bx_k \tag{3.13}$$

Suppose that $x_k = p_k$ is a particular solution of Equation 3.12.

a. If $q_k$ is any solution of Equation 3.13, show that $q_k + p_k$ is a solution of Equation 3.12.

b. Show that every solution of Equation 3.12 arises as in (a) as the sum of a solution of Equation 3.13 plus the particular solution $p_k$ of Equation 3.12.

## 3.5 An Application to Systems of Differential Equations

A function $f$ of a real variable is said to be **differentiable** if its derivative exists and, in this case, we let $f'$ denote the derivative. If $f$ and $g$ are differentiable functions, a system

$$\begin{aligned} f' &= 3f + 5g \\ g' &= -f + 2g \end{aligned}$$

is called a *system of first order differential equations,* or a *differential system* for short. Solving many practical problems often comes down to finding sets of functions that satisfy such a system (often involving more than two functions). In this section we show how diagonalization can help. Of course an acquaintance with calculus is required.

### The Exponential Function

The simplest differential system is the following single equation:

$$f' = af \text{ where } a \text{ is constant} \tag{3.14}$$

It is easily verified that $f(x) = e^{ax}$ is one solution; in fact, Equation 3.14 is simple enough for us to find *all* solutions. Suppose that $f$ is any solution, so that $f'(x) = af(x)$ for all $x$. Consider the new function $g$ given by $g(x) = f(x)e^{-ax}$. Then the product rule of differentiation gives

$$\begin{aligned} g'(x) &= f(x)\left[-ae^{-ax}\right] + f'(x)e^{-ax} \\ &= -af(x)e^{-ax} + [af(x)]\, e^{-ax} \\ &= 0 \end{aligned}$$

for all $x$. Hence the function $g(x)$ has zero derivative and so must be a constant, say $g(x) = c$. Thus $c = g(x) = f(x)e^{-ax}$, that is

$$f(x) = ce^{ax}$$

In other words, every solution $f(x)$ of Equation 3.14 is just a scalar multiple of $e^{ax}$. Since every such scalar multiple is easily seen to be a solution of Equation 3.14, we have proved

**Theorem 3.5.1**

*The set of solutions to $f' = af$ is $\{ce^{ax} \mid c \text{ any constant}\} = \mathbb{R}e^{ax}$.*

Remarkably, this result together with diagonalization enables us to solve a wide variety of differential systems.

**Example 3.5.1**

Assume that the number $n(t)$ of bacteria in a culture at time $t$ has the property that the rate of change of $n$ is proportional to $n$ itself. If there are $n_0$ bacteria present when $t = 0$, find the number at time $t$.

**Solution.** Let $k$ denote the proportionality constant. The rate of change of $n(t)$ is its time-derivative

$n'(t)$, so the given relationship is $n'(t) = kn(t)$. Thus Theorem 3.5.1 shows that all solutions $n$ are given by $n(t) = ce^{kt}$, where $c$ is a constant. In this case, the constant $c$ is determined by the requirement that there be $n_0$ bacteria present when $t = 0$. Hence $n_0 = n(0) = ce^{k0} = c$, so

$$n(t) = n_0 e^{kt}$$

gives the number at time $t$. Of course the constant $k$ depends on the strain of bacteria.

The condition that $n(0) = n_0$ in Example 3.5.1 is called an **initial condition** or a **boundary condition** and serves to select one solution from the available solutions.

## General Differential Systems

Solving a variety of problems, particularly in science and engineering, comes down to solving a system of linear differential equations. Diagonalization enters into this as follows. The general problem is to find differentiable functions $f_1, f_2, \dots, f_n$ that satisfy a system of equations of the form

$$\begin{array}{l} f_1' = a_{11}f_1 + a_{12}f_2 + \cdots + a_{1n}f_n \\ f_2' = a_{21}f_1 + a_{22}f_2 + \cdots + a_{2n}f_n \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad\quad\quad\quad \vdots \\ f_n' = a_{n1}f_1 + a_{n2}f_2 + \cdots + a_{nn}f_n \end{array}$$

where the $a_{ij}$ are constants. This is called a **linear system of differential equations** or simply a **differential system**. The first step is to put it in matrix form. Write

$$\mathbf{f} = \left[ \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_n \end{array} \right] \quad \mathbf{f}' = \left[ \begin{array}{c} f_1' \\ f_2' \\ \vdots \\ f_n' \end{array} \right] \quad A = \left[ \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right]$$

Then the system can be written compactly using matrix multiplication:

$$\mathbf{f}' = A\mathbf{f}$$

Hence, given the matrix $A$, the problem is to find a column $\mathbf{f}$ of differentiable functions that satisfies this condition. This can be done if $A$ is diagonalizable. Here is an example.

**Example 3.5.2**

Find a solution to the system

$$\begin{array}{rcl} f_1' & = & f_1 + 3f_2 \\ f_2' & = & 2f_1 + 2f_2 \end{array}$$

that satisfies $f_1(0) = 0$, $f_2(0) = 5$.

**Solution.** This is $\mathbf{f}' = A\mathbf{f}$, where $\mathbf{f} = \left[\begin{array}{c} f_1 \\ f_2 \end{array}\right]$ and $A = \left[\begin{array}{cc} 1 & 3 \\ 2 & 2 \end{array}\right]$. The reader can verify that $c_A(x) = (x-4)(x+1)$, and that $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} 3 \\ -2 \end{array}\right]$ are eigenvectors corresponding to the eigenvalues 4 and $-1$, respectively. Hence the diagonalization algorithm gives $P^{-1}AP = \left[\begin{array}{cc} 4 & 0 \\ 0 & -1 \end{array}\right]$, where $P = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \\ 1 & -2 \end{array}\right]$. Now consider new functions $g_1$ and $g_2$ given by $\mathbf{f} = P\mathbf{g}$ (equivalently, $\mathbf{g} = P^{-1}\mathbf{f}$ ), where $\mathbf{g} = \left[\begin{array}{c} g_1 \\ g_2 \end{array}\right]$ Then

$$\left[\begin{array}{c} f_1 \\ f_2 \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \\ 1 & -2 \end{array}\right]\left[\begin{array}{c} g_1 \\ g_2 \end{array}\right] \quad \text{that is,} \quad \begin{array}{l} f_1 = g_1 + 3g_2 \\ f_2 = g_1 - 2g_2 \end{array}$$

Hence $f_1' = g_1' + 3g_2'$ and $f_2' = g_1' - 2g_2'$ so that

$$\mathbf{f}' = \left[\begin{array}{c} f_1' \\ f_2' \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \\ 1 & -2 \end{array}\right]\left[\begin{array}{c} g_1' \\ g_2' \end{array}\right] = P\mathbf{g}'$$

If this is substituted in $\mathbf{f}' = A\mathbf{f}$, the result is $P\mathbf{g}' = AP\mathbf{g}$, whence

$$\mathbf{g}' = P^{-1}AP\mathbf{g}$$

But this means that

$$\left[\begin{array}{c} g_1' \\ g_2' \end{array}\right] = \left[\begin{array}{cc} 4 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{c} g_1 \\ g_2 \end{array}\right], \quad \text{so} \quad \begin{array}{l} g_1' = 4g_1 \\ g_2' = -g_2 \end{array}$$

Hence Theorem 3.5.1 gives $g_1(x) = ce^{4x}$, $g_2(x) = de^{-x}$, where $c$ and $d$ are constants. Finally, then,

$$\left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array}\right] = P\left[\begin{array}{c} g_1(x) \\ g_2(x) \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \\ 1 & -2 \end{array}\right]\left[\begin{array}{c} ce^{4x} \\ de^{-x} \end{array}\right] = \left[\begin{array}{c} ce^{4x} + 3de^{-x} \\ ce^{4x} - 2de^{-x} \end{array}\right]$$

so the *general solution* is

$$\begin{array}{rcl} f_1(x) & = & ce^{4x} + 3de^{-x} \\ f_2(x) & = & ce^{4x} - 2de^{-x} \end{array} \quad c \text{ and } d \text{ constants}$$

It is worth observing that this can be written in matrix form as

$$\left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array}\right] = c\left[\begin{array}{c} 1 \\ 1 \end{array}\right]e^{4x} + d\left[\begin{array}{c} 3 \\ -2 \end{array}\right]e^{-x}$$

That is,

$$\mathbf{f}(x) = c\mathbf{x}_1e^{4x} + d\mathbf{x}_2e^{-x}$$

This form of the solution works more generally, as will be shown.
Finally, the requirement that $f_1(0) = 0$ and $f_2(0) = 5$ in this example determines the constants $c$ and $d$:

$$0 = f_1(0) = ce^0 + 3de^0 = c + 3d$$

$$5 = f_2(0) = ce^0 - 2de^0 = c - 2d$$

These equations give $c = 3$ and $d = -1$, so

$$f_1(x) = 3e^{4x} - 3e^{-x}$$
$$f_2(x) = 3e^{4x} + 2e^{-x}$$

satisfy all the requirements.

The technique in this example works in general.

**Theorem 3.5.2**

*Consider a linear system*

$$\mathbf{f}' = A\mathbf{f}$$

*of differential equations, where $A$ is an $n \times n$ diagonalizable matrix. Let $P^{-1}AP$ be diagonal, where $P$ is given in terms of its columns*

$$P = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n]$$

*and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ are eigenvectors of $A$. If $\mathbf{x}_i$ corresponds to the eigenvalue $\lambda_i$ for each $i$, then every solution $\mathbf{f}$ of $\mathbf{f}' = A\mathbf{f}$ has the form*

$$\mathbf{f}(x) = c_1\mathbf{x}_1 e^{\lambda_1 x} + c_2\mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n\mathbf{x}_n e^{\lambda_n x}$$

*where $c_1, c_2, \ldots, c_n$ are arbitrary constants.*

**Proof.** By Theorem 3.3.4, the matrix $P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{array}\right]$ is invertible and

$$P^{-1}AP = \left[\begin{array}{cccc} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{array}\right]$$

As in Example 3.5.2, write $\mathbf{f} = \left[\begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_n \end{array}\right]$ and define $\mathbf{g} = \left[\begin{array}{c} g_1 \\ g_2 \\ \vdots \\ g_n \end{array}\right]$ by $\mathbf{g} = P^{-1}\mathbf{f}$; equivalently, $\mathbf{f} = P\mathbf{g}$. If $P = \left[p_{ij}\right]$, this gives

$$f_i = p_{i1}g_1 + p_{i2}g_2 + \cdots + p_{in}g_n$$

Since the $p_{ij}$ are constants, differentiation preserves this relationship:

$$f_i' = p_{i1}g_1' + p_{i2}g_2' + \cdots + p_{in}g_n'$$

so $\mathbf{f}' = P\mathbf{g}'$. Substituting this into $\mathbf{f}' = A\mathbf{f}$ gives $P\mathbf{g}' = AP\mathbf{g}$. But then left multiplication by $P^{-1}$ gives

$\mathbf{g}' = P^{-1}AP\mathbf{g}$, so the original system of equations $\mathbf{f}' = A\mathbf{f}$ for $\mathbf{f}$ becomes much simpler in terms of $\mathbf{g}$:

$$\begin{bmatrix} g_1' \\ g_2' \\ \vdots \\ g_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

Hence $g_i' = \lambda_i g_i$ holds for each $i$, and Theorem 3.5.1 implies that the only solutions are

$$g_i(x) = c_i e^{\lambda_i x} \quad c_i \text{ some constant}$$

Then the relationship $\mathbf{f} = P\mathbf{g}$ gives the functions $f_1, f_2, \dots, f_n$ as follows:

$$\mathbf{f}(x) = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n] \begin{bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ \vdots \\ c_n e^{\lambda_n x} \end{bmatrix} = c_1 \mathbf{x}_1 e^{\lambda_1 x} + c_2 \mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n \mathbf{x}_n e^{\lambda_n x}$$

This is what we wanted. □

The theorem shows that *every* solution to $\mathbf{f}' = A\mathbf{f}$ is a linear combination

$$\mathbf{f}(x) = c_1 \mathbf{x}_1 e^{\lambda_1 x} + c_2 \mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n \mathbf{x}_n e^{\lambda_n x}$$

where the coefficients $c_i$ are arbitrary. Hence this is called the **general solution** to the system of differential equations. In most cases the solution functions $f_i(x)$ are required to satisfy boundary conditions, often of the form $f_i(a) = b_i$, where $a, b_1, \dots, b_n$ are prescribed numbers. These conditions determine the constants $c_i$. The following example illustrates this and displays a situation where one eigenvalue has multiplicity greater than 1.

**Example 3.5.3**

Find the general solution to the system

$$\begin{aligned} f_1' &= 5f_1 + 8f_2 + 16f_3 \\ f_2' &= 4f_1 + f_2 + 8f_3 \\ f_3' &= -4f_1 - 4f_2 - 11f_3 \end{aligned}$$

Then find a solution satisfying the boundary conditions $f_1(0) = f_2(0) = f_3(0) = 1$.

**Solution.** The system has the form $\mathbf{f}' = A\mathbf{f}$, where $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. In this case $c_A(x) = (x+3)^2(x-1)$ and eigenvectors corresponding to the eigenvalues $-3$, $-3$, and 1 are, respectively,

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Hence, by Theorem 3.5.2, the general solution is

$$\mathbf{f}(x) = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-3x} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-3x} + c_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} e^{x}, \qquad c_i \text{ constants.}$$

The boundary conditions $f_1(0) = f_2(0) = f_3(0) = 1$ determine the constants $c_i$.

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{f}(0) &= c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

The solution is $c_1 = -3$, $c_2 = 5$, $c_3 = 4$, so the required specific solution is

$$\begin{aligned} f_1(x) &= -7e^{-3x} + 8e^{x} \\ f_2(x) &= -3e^{-3x} + 4e^{x} \\ f_3(x) &= \quad 5e^{-3x} - 4e^{x} \end{aligned}$$

## Exercises for 3.5

**Exercise 3.5.1** Use Theorem 3.5.1 to find the general solution to each of the following systems. Then find a specific solution satisfying the given boundary condition.

a. $f_1' = 2f_1 + 4f_2, \ f_1(0) = 0$
$f_2' = 3f_1 + 3f_2, \ f_2(0) = 1$

b. $f_1' = -f_1 + 5f_2, \ f_1(0) = \ \ 1$
$f_2' \ = f_1 + 3f_2, \ f_2(0) = -1$

c. $f_1' = \quad\quad 4f_2 + 4f_3$
$f_2' = \ \ f_1 + f_2 - 2f_3$
$f_3' = -f_1 + f_2 + 4f_3$
$f_1(0) = f_2(0) = f_3(0) = 1$

d. $f_1' = 2f_1 + \ \ f_2 + 2f_3$
$f_2' = 2f_1 + 2f_2 - 2f_3$
$f_3' = 3f_1 + \ \ f_2 + \ \ f_3$
$f_1(0) = f_2(0) = f_3(0) = 1$

**Exercise 3.5.2** Show that the solution to $f' = af$ satisfying $f(x_0) = k$ is $f(x) = ke^{a(x-x_0)}$.

**Exercise 3.5.3** A radioactive element decays at a rate proportional to the amount present. Suppose an initial mass of 10 g decays to 8 g in 3 hours.

a. Find the mass $t$ hours later.

b. Find the half-life of the element—the time taken to decay to half its mass.

**Exercise 3.5.4** The population $N(t)$ of a region at time $t$ increases at a rate proportional to the population. If the population doubles every 5 years and is 3 million initially, find $N(t)$.

**Exercise 3.5.5** Let $A$ be an invertible diagonalizable $n \times n$ matrix and let $\mathbf{b}$ be an $n$-column of constant functions. We can solve the system $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$ as follows:

a. If $\mathbf{g}$ satisfies $\mathbf{g}' = A\mathbf{g}$ (using Theorem 3.5.2), show that $\mathbf{f} = \mathbf{g} - A^{-1}\mathbf{b}$ is a solution to $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$.

b. Show that every solution to $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$ arises as in (a) for some solution $\mathbf{g}$ to $\mathbf{g}' = A\mathbf{g}$.

**Exercise 3.5.6** Denote the second derivative of $f$ by $f'' = (f')'$. Consider the second order differential equation

$$f'' - a_1 f' - a_2 f = 0, \quad a_1 \text{ and } a_2 \text{ real numbers} \quad (3.15)$$

a. If $f$ is a solution to Equation 3.15 let $f_1 = f$ and $f_2 = f' - a_1 f$. Show that
$$\begin{cases} f_1' = a_1 f_1 + f_2 \\ f_2' = a_2 f_1 \end{cases},$$
that is $\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

b. Conversely, if $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a solution to the system in (a), show that $f_1$ is a solution to Equation 3.15.

**Exercise 3.5.7** Writing $f''' = (f'')'$, consider the third order differential equation

$$f''' - a_1 f'' - a_2 f' - a_3 f = 0$$

where $a_1$, $a_2$, and $a_3$ are real numbers. Let $f_1 = f$, $f_2 = f' - a_1 f$ and $f_3 = f'' - a_1 f' - a_2 f''$.

a. Show that $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is a solution to the system
$$\begin{cases} f_1' = a_1 f_1 + f_2 \\ f_2' = a_2 f_1 + f_3, \\ f_3' = a_3 f_1 \end{cases}$$
that is $\begin{bmatrix} f_1' \\ f_2' \\ f_3' \end{bmatrix} = \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

b. Show further that if $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is any solution to this system, then $f = f_1$ is a solution to Equation 3.15.

*Remark*. A similar construction casts every linear differential equation of order $n$ (with constant coefficients) as an $n \times n$ linear system of first order equations. However, the matrix need not be diagonalizable, so other methods have been developed.

## 3.6 Proof of the Cofactor Expansion Theorem

Recall that our definition of the term *determinant* is inductive: The determinant of any $1 \times 1$ matrix is defined first; then it is used to define the determinants of $2 \times 2$ matrices. Then that is used for the $3 \times 3$ case, and so on. The case of a $1 \times 1$ matrix $[a]$ poses no problem. We simply define

$$\det [a] = a$$

as in Section 3.1. Given an $n \times n$ matrix $A$, define $A_{ij}$ to be the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting row $i$ and column $j$. Now assume that the determinant of any $(n-1) \times (n-1)$ matrix has been defined. Then the determinant of $A$ is *defined* to be

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{21} \det A_{21} + \cdots + (-1)^{n+1} a_{n1} \det A_{n1} \\ &= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1} \end{aligned}$$

where summation notation has been introduced for convenience.[16] Observe that, in the terminology of Section 3.1, this is just the cofactor expansion of $\det A$ along the first column, and that $(-1)^{i+j} \det A_{ij}$ is the $(i,\ j)$-cofactor (previously denoted as $c_{ij}(A)$).[17] To illustrate the definition, consider the $2 \times 2$ matrix

[16]Summation notation is a convenient shorthand way to write sums of similar expressions. For example $a_1 + a_2 + a_3 + a_4 = \sum_{i=1}^{4} a_i$, $a_5 b_5 + a_6 b_6 + a_7 b_7 + a_8 b_8 = \sum_{k=5}^{8} a_k b_k$, and $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{j=1}^{5} j^2$.

[17]Note that we used the expansion along row 1 at the beginning of Section 3.1. The column 1 expansion definition is more convenient here.

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then the definition gives

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \det [a_{22}] - a_{21} \det [a_{12}] = a_{11}a_{22} - a_{21}a_{12}$$

and this is the same as the definition in Section 3.1.

Of course, the task now is to use this definition to *prove* that the cofactor expansion along *any* row or column yields $\det A$ (this is Theorem 3.1.1). The proof proceeds by first establishing the properties of determinants stated in Theorem 3.1.2 but for *rows* only (see Lemma 3.6.2). This being done, the full proof of Theorem 3.1.1 is not difficult. The proof of Lemma 3.6.2 requires the following preliminary result.

**Lemma 3.6.1**

*Let $A$, $B$, and $C$ be $n \times n$ matrices that are identical except that the $p$th row of $A$ is the sum of the $p$th rows of $B$ and $C$. Then*

$$\det A = \det B + \det C$$

**Proof.** We proceed by induction on $n$, the cases $n = 1$ and $n = 2$ being easily checked. Consider $a_{i1}$ and $A_{i1}$:

Case 1: If $i \neq p$,

$$a_{i1} = b_{i1} = c_{i1} \quad \text{and} \quad \det A_{i1} = \det B_{i1} = \det C_{i1}$$

by induction because $A_{i1}$, $B_{i1}$, $C_{i1}$ are identical except that one row of $A_{i1}$ is the sum of the corresponding rows of $B_{i1}$ and $C_{i1}$.

Case 2: If $i = p$,

$$a_{p1} = b_{p1} + c_{p1} \quad \text{and} \quad A_{p1} = B_{p1} = C_{p1}$$

Now write out the defining sum for $\det A$, splitting off the $p$th term for special attention.

$$\begin{aligned} \det A &= \sum_{i \neq p} a_{i1}(-1)^{i+1} \det A_{i1} + a_{p1}(-1)^{p+1} \det A_{p1} \\ &= \sum_{i \neq p} a_{i1}(-1)^{i+1} [\det B_{i1} + \det B_{i1}] + (b_{p1} + c_{p1})(-1)^{p+1} \det A_{p1} \end{aligned}$$

where $\det A_{i1} = \det B_{i1} + \det C_{i1}$ by induction. But the terms here involving $B_{i1}$ and $b_{p1}$ add up to $\det B$ because $a_{i1} = b_{i1}$ if $i \neq p$ and $A_{p1} = B_{p1}$. Similarly, the terms involving $C_{i1}$ and $c_{p1}$ add up to $\det C$. Hence $\det A = \det B + \det C$, as required. □

**Lemma 3.6.2**

*Let $A = [a_{ij}]$ denote an $n \times n$ matrix.*

1. *If $B = [b_{ij}]$ is formed from $A$ by multiplying a row of $A$ by a number $u$, then $\det B = u \det A$.*
2. *If $A$ contains a row of zeros, then $\det A = 0$.*
3. *If $B = [b_{ij}]$ is formed by interchanging two rows of $A$, then $\det B = -\det A$.*
4. *If $A$ contains two identical rows, then $\det A = 0$.*

5. *If* $B = [b_{ij}]$ *is formed by adding a multiple of one row of* $A$ *to a different row, then* $\det B = \det A$.

**Proof.** For later reference the defining sums for $\det A$ and $\det B$ are as follows:

$$\det A = \sum_{i=1}^{n} a_{i1}(-1)^{i+1} \det A_{i1} \tag{3.16}$$

$$\det B = \sum_{i=1}^{n} b_{i1}(-1)^{i+1} \det B_{i1} \tag{3.17}$$

*Property 1.* The proof is by induction on $n$, the cases $n = 1$ and $n = 2$ being easily verified. Consider the $i$th term in the sum 3.17 for $\det B$ where $B$ is the result of multiplying row $p$ of $A$ by $u$.

a. If $i \neq p$, then $b_{i1} = a_{i1}$ and $\det B_{i1} = u \det A_{i1}$ by induction because $B_{i1}$ comes from $A_{i1}$ by multiplying a row by $u$.

b. If $i = p$, then $b_{p1} = ua_{p1}$ and $B_{p1} = A_{p1}$.

In either case, each term in Equation 3.17 is $u$ times the corresponding term in Equation 3.16, so it is clear that $\det B = u \det A$.

*Property 2.* This is clear by property 1 because the row of zeros has a common factor $u = 0$.

*Property 3.* Observe first that it suffices to prove property 3 for interchanges of adjacent rows. (Rows $p$ and $q$ $(q > p)$ can be interchanged by carrying out $2(q - p) - 1$ adjacent changes, which results in an *odd* number of sign changes in the determinant.) So suppose that rows $p$ and $p + 1$ of $A$ are interchanged to obtain $B$. Again consider the $i$th term in Equation 3.17.

a. If $i \neq p$ and $i \neq p + 1$, then $b_{i1} = a_{i1}$ and $\det B_{i1} = -\det A_{i1}$ by induction because $B_{i1}$ results from interchanging adjacent rows in $A_{i1}$. Hence the $i$th term in Equation 3.17 is the negative of the $i$th term in Equation 3.16. Hence $\det B = -\det A$ in this case.

b. If $i = p$ or $i = p + 1$, then $b_{p1} = a_{p+1,\,1}$ and $B_{p1} = A_{p+1,\,1}$, whereas $b_{p+1,\,1} = a_{p1}$ and $B_{p+1,\,1} = A_{p1}$. Hence terms $p$ and $p + 1$ in Equation 3.17 are

$$b_{p1}(-1)^{p+1} \det B_{p1} = -a_{p+1,\,1}(-1)^{(p+1)+1} \det (A_{p+1,\,1})$$

$$b_{p+1,\,1}(-1)^{(p+1)+1} \det B_{p+1,\,1} = -a_{p1}(-1)^{p+1} \det (A_{p1})$$

This means that terms $p$ and $p + 1$ in Equation 3.17 are the same as these terms in Equation 3.16, except that the order is reversed and the signs are changed. Thus the sum 3.17 is the negative of the sum 3.16; that is, $\det B = -\det A$.

*Property 4.* If rows $p$ and $q$ in $A$ are identical, let $B$ be obtained from $A$ by interchanging these rows. Then $B = A$ so $\det A = \det B$. But $\det B = -\det A$ by property 3 so $\det A = -\det A$. This implies that $\det A = 0$.

*Property 5.* Suppose $B$ results from adding $u$ times row $q$ of $A$ to row $p$. Then Lemma 3.6.1 applies to $B$ to show that $\det B = \det A + \det C$, where $C$ is obtained from $A$ by replacing row $p$ by $u$ times row $q$. It now follows from properties 1 and 4 that $\det C = 0$ so $\det B = \det A$, as asserted. □

These facts are enough to enable us to prove Theorem 3.1.1. For convenience, it is restated here in the notation of the foregoing lemmas. The only difference between the notations is that the $(i, j)$-cofactor of an $n \times n$ matrix $A$ was denoted earlier by

$$c_{ij}(A) = (-1)^{i+j} \det A_{ij}$$

**Theorem 3.6.1**

*If $A = \left[a_{ij}\right]$ is an $n \times n$ matrix, then*

1. $\det A = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}$ *(cofactor expansion along column $j$).*
2. $\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}$ *(cofactor expansion along row $i$).*

*Here $A_{ij}$ denotes the matrix obtained from $A$ by deleting row $i$ and column $j$.*

**Proof.** Lemma 3.6.2 establishes the truth of Theorem 3.1.2 for *rows*. With this information, the arguments in Section 3.2 proceed exactly as written to establish that $\det A = \det A^T$ holds for any $n \times n$ matrix $A$. Now suppose $B$ is obtained from $A$ by interchanging two columns. Then $B^T$ is obtained from $A^T$ by interchanging two rows so, by property 3 of Lemma 3.6.2,

$$\det B = \det B^T = -\det A^T = -\det A$$

Hence property 3 of Lemma 3.6.2 holds for *columns* too.

This enables us to prove the cofactor expansion for columns. Given an $n \times n$ matrix $A = \left[a_{ij}\right]$, let $B = \left[b_{ij}\right]$ be obtained by moving column $j$ to the left side, using $j-1$ interchanges of adjacent columns. Then $\det B = (-1)^{j-1} \det A$ and, because $B_{i1} = A_{ij}$ and $b_{i1} = a_{ij}$ for all $i$, we obtain

$$\begin{aligned}
\det A = (-1)^{j-1} \det B &= (-1)^{j-1} \sum_{i=1}^{n} b_{i1}(-1)^{i+1} \det B_{i1} \\
&= \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}
\end{aligned}$$

This is the cofactor expansion of $\det A$ along column $j$.

Finally, to prove the row expansion, write $B = A^T$. Then $B_{ij} = (A_{ij}^T)$ and $b_{ij} = a_{ji}$ for all $i$ and $j$. Expanding $\det B$ along column $j$ gives

$$\begin{aligned}
\det A = \det A^T = \det B &= \sum_{i=1}^{n} b_{ij}(-1)^{i+j} \det B_{ij} \\
&= \sum_{i=1}^{n} a_{ji}(-1)^{j+i} \det \left[(A_{ji}^T)\right] = \sum_{i=1}^{n} a_{ji}(-1)^{j+i} \det A_{ji}
\end{aligned}$$

This is the required expansion of $\det A$ along row $j$. □

## Exercises for 3.6

**Exercise 3.6.1** Prove Lemma 3.6.1 for columns.

**Exercise 3.6.2** Verify that interchanging rows $p$ and $q$ $(q > p)$ can be accomplished using $2(q-p)-1$ adjacent interchanges.

**Exercise 3.6.3** If $u$ is a number and $A$ is an $n \times n$ matrix, prove that $\det(uA) = u^n \det A$ by induction on $n$, using only the definition of $\det A$.

## Supplementary Exercises for Chapter 3

**Exercise 3.1** Show that

$$\det \left[ \begin{array}{ccc} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{array} \right] = (1+x^3) \det \left[ \begin{array}{ccc} a & b & c \\ p & q & r \\ u & v & w \end{array} \right]$$

**Exercise 3.2**

a. Show that $(A_{ij})^T = (A^T)_{ji}$ for all $i$, $j$, and all square matrices $A$.

b. Use (a) to prove that $\det A^T = \det A$. [*Hint*: Induction on $n$ where $A$ is $n \times n$.]

**Exercise 3.3** Show that $\det \left[ \begin{array}{cc} 0 & I_n \\ I_m & 0 \end{array} \right] = (-1)^{nm}$ for all $n \geq 1$ and $m \geq 1$.

**Exercise 3.4** Show that

$$\det \left[ \begin{array}{ccc} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{array} \right] = (b-a)(c-a)(c-b)(a+b+c)$$

**Exercise 3.5** Let $A = \left[ \begin{array}{c} R_1 \\ R_2 \end{array} \right]$ be a $2 \times 2$ matrix with rows $R_1$ and $R_2$. If $\det A = 5$, find $\det B$ where

$$B = \left[ \begin{array}{c} 3R_1 + 2R_3 \\ 2R_1 + 5R_2 \end{array} \right]$$

**Exercise 3.6** Let $A = \left[ \begin{array}{cc} 3 & -4 \\ 2 & -3 \end{array} \right]$ and let $\mathbf{v}_k = A^k \mathbf{v}_0$ for each $k \geq 0$.

a. Show that $A$ has no dominant eigenvalue.

b. Find $\mathbf{v}_k$ if $\mathbf{v}_0$ equals:

i. $\left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$

ii. $\left[ \begin{array}{c} 2 \\ 1 \end{array} \right]$

iii. $\left[ \begin{array}{c} x \\ y \end{array} \right] \neq \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$ or $\left[ \begin{array}{c} 2 \\ 1 \end{array} \right]$