Exercise 2.2.10 In each case either show that the statement is true, or give an example showing that it is false.

a. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

b. If $A\mathbf{x}$ has a zero entry, then A has a row of zeros.

c. If $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} \neq \mathbf{0}$, then $A = 0$.

d. Every linear combination of vectors in $\mathbb{R}^n$ can be written in the form $A\mathbf{x}$.

e. If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns, and if $\mathbf{b} = 3\mathbf{a}_1 - 2\mathbf{a}_2$, then the system $A\mathbf{x} = \mathbf{b}$ has a solution.

If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns, and if the system $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{b} = s\mathbf{a}_1 + t\mathbf{a}_2$ for some s, t.

g. If A is $m \times n$ and $m < n$, then $A\mathbf{x} = \mathbf{b}$ has a solution for every column $\mathbf{b}$.

h. If $A\mathbf{x} = \mathbf{b}$ has a solution for some column $\mathbf{b}$, then it has a solution for every column $\mathbf{b}$.

i. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution to $A\mathbf{x} = \mathbf{0}$.

j. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns. If $\mathbf{a}_3 = s\mathbf{a}_1 + t\mathbf{a}_2$, then $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} s \\ t \\ -1 \end{bmatrix}$.

Statement:

If A is a matrix and $\vec{x}$ is a vector and $A\vec{x}$ has a zero entry, THEN A has a row of zeros.

False:

Counterexample 1: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1\cdot 2 + 2\cdot -1 \\ 2\cdot 2 + -1\cdot 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

even though A has no row of zeros.

Counterexample 2:

Try A w/ dim 3×2

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Want $\vec{x}$ so that $\underset{3\times 2}{A} \, \underset{2\times 1}{\vec{x}} = \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$

Want $\vec{x}$ such that

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1x_1 + 2x_2 \\ 0 + x_2 \\ x_1 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 1x_1 + 2x_2 &= 0 \\ 1x_2 &= 0 \\ 1x_1 \quad &= 0 \end{aligned}\right\} \text{A system of linear equations}$$

Write the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

Row reduce:

$$\begin{array}{r} R_1 \\ R_2 \\ -1R_1 + R_3 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right] \longrightarrow \begin{array}{r} R_1 \\ R_2 \\ -\frac{1}{2}R_3 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow \begin{array}{r} R_1 \\ R_2 \\ -R_2 + R_3 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_2 = 0$

$\boxed{x_2 = 0}$

$\boxed{x_1 = 0}$

Our counterexample 2: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ even though A has no row of zeros.

Solution manual:

10. b. False. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has a zero entry, but $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no zero row.

d. True. The linear combination $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$ equals $A\mathbf{x}$ where, by Theorem 2.2.1, $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ is the matrix with these vectors $\mathbf{a}_i$ as its columns.

False. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ then $A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and this is not a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ because it is not a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

h. False. If $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, there is a solution $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ for $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. But there is no solution for $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Indeed, if $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $x - y + z = 1$ and $-x + y - z = 0$. This is impossible.

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b. If $A\mathbf{x}$ has a zero entry, then A has a row of zeros.

c. If $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} \neq \mathbf{0}$, then $A = 0$.

d. Every linear combination of vectors in $\mathbb{R}^n$ can be written in the form $A\mathbf{x}$.

True

e. If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns, and if $\mathbf{b} = 3\mathbf{a}_1 - 2\mathbf{a}_2$, then the system $A\mathbf{x} = \mathbf{b}$ has a solution.

If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns, and if the system $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{b} = s\mathbf{a}_1 + t\mathbf{a}_2$ for some s, t.

g. If A is $m \times n$ and $m < n$, then $A\mathbf{x} = \mathbf{b}$ has a solution for every column $\mathbf{b}$.

h. If $A\mathbf{x} = \mathbf{b}$ has a solution for some column $\mathbf{b}$, then it has a solution for every column $\mathbf{b}$.

i. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution to $A\mathbf{x} = \mathbf{0}$.

j. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns. If $\mathbf{a}_3 = s\mathbf{a}_1 + t\mathbf{a}_2$, then $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} s \\ t \\ -1 \end{bmatrix}$.

Statement: Every linear combination $r_1 v_1 + r_2 v_2 + r_3 v_3 + \dots + r_k v_k$ of $v_1, v_2, \dots, v_k$ can be written in the form

$$\begin{bmatrix} \text{matrix} \end{bmatrix} \begin{bmatrix} \text{a} \\ \text{vector} \end{bmatrix}.$$

E.g. $2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 5 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 8 \\ 9 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ 9 \end{bmatrix}$.

Can I write this as $\underbrace{A}_{\text{matrix}} \underbrace{\vec{x}}_{\text{vector}}$?

$$2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 5 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \underbrace{\begin{bmatrix} 2.1 + 4.5 + 7.8 \\ 2.3 + 4.6 + 7.9 \end{bmatrix}}_{2\times 1 \text{ vector}}$$

$$\begin{bmatrix} 2.1 + 4.5 + 7.8 \\ 2.3 + 4.6 + 7.9 \end{bmatrix} \stackrel{?}{=} A\vec{x} = \begin{bmatrix} 1 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

$$r_1 v_1 + r_2 v_2 + r_3 v_3 + \dots + r_k v_k = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ v_k \end{bmatrix}$$

6 Question (2D transformation)

Let $M := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

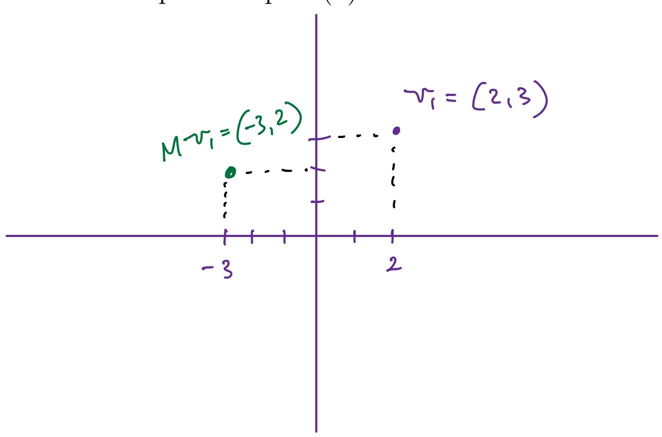
a. Compute the product $M \begin{bmatrix} x \\ y \end{bmatrix}$.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

b. Compute the following vectors:

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}$ $\qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \boxed{}$ $\qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \end{bmatrix} = \boxed{}$

c. Plot the points $v_1 = (2, 3)$, $v_2 = (4, 6)$, $v_3 = (6, 9)$ in Cartesian coordinates. On the same graph, plot the points corresponding to the vectors computed in part (b).



d. Describe what the matrix M does to the points v_1, v_2, v_3. Hint: Use phrases like "rotation by . . . degrees" or "reflection across . . . line".

Rotation by - - - - - degrees

Clockwise / counterclockwise ?

e. What does the matrix M^2 do to the points v_1, v_2, v_3? $\quad M^2 = MM = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$

Plot $M^2 v_1$, $M^2 v_2$, $M^2 v_3$.

Rotation by - - - - - degrees

Clockwise / counterclockwise ?

7 Question (2D transformation)

(This question is a 2D preview of a future topic.)

Let $B := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(a) Compute the product $B \begin{bmatrix} x \\ y \end{bmatrix}$.

(b) Compute the following vectors:

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boxed{}$ $\quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \boxed{}$ $\quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \boxed{}$

(c) Plot the points $v_1 = (2, 3)$, $v_2 = (4, 6)$, $v_3 = (6, 9)$ in Cartesian coordinates. On the same graph, plot the points corresponding to the vectors computed in part (b)

(d) Describe what the matrix B does to the points v_1, v_2, v_3. Hint: Use phrases like "rotation by ... degrees" or "reflection across ... line". Specify the equation for this line!

(e) What does the matrix B^2 do to the points v_1, v_2, v_3?