Q1 Key

(Source Lecture 1b Exercise 2)

a)
$$x + y + z = 3$$
$$x + y + 2z = 4$$
$$y + 2z = 2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & -1 & -2 & -4 \\ 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \mapsto R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\left.\begin{array}{r} x + y + z = 3 \\ -z = -1 \\ y + 2z = 2 \end{array}\right\}$$

$x + 0 + 1 = 3 \Rightarrow \boxed{x = 2}$

$\boxed{z = 1}$

$y + 2(1) = 2 \Rightarrow \boxed{y = 0}$

Solution:
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

b. Write down a matrix multiplication which you can perform to verify your solution

$$* \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

* Perform the matrix multiplication.

$$\begin{bmatrix} 1.2 + 0 + 1 \\ 1.2 + 0 + 2.1 \\ 0 + 0 + 2.1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \checkmark$$

Q 2 Key (Source: Lecture 2a, 2b)

a)

$M :=$

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \xrightarrow{R_2 \mapsto -2R_1 + R_2} \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \xrightarrow{R_3 \mapsto -R_1 + R_3} \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \mapsto \frac{1}{3}R_2} \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

one of the many REF matrices equivalent to M

b) The above REF matrix has two leading 1s, so rank $(M) = 2$.

c) For example,
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 6 \\ 0 & 1 & 6 \\ 0 & 1 & 6 \end{bmatrix}$$
are all 4x3 matrices with rank 2.

Q3 Key (Source: Lecture 2a)

Find solutions to the system

$$\begin{cases} a - 2b \quad + d = 2 \\ \qquad\quad c - 2d = 1 \end{cases}$$

Think: $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$

The 2nd & 4th columns (left of bar) have no leading 1s.

a. Two columns have no leading 1. So, two parameters are needed to describe the solutions.

b. Let $\boxed{b = t, \quad d = s}$

Then $c - 2d = 1 \Rightarrow c - 2s = 1 \Rightarrow \boxed{c = 1 + 2s}$

$a - 2b + d = 2 \Rightarrow a - 2t + s = 2 \Rightarrow \boxed{a = 2 + 2t - s}$

The solutions are

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 + 2t - s \\ t \\ 1 + 2s \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

for all numbers t and s.

c. The dimension is 2

d. The shape is a plane

(source: Lecture 9a)

Q4 Key

a) (Source: Lecture 2a)

$$\left[\begin{array}{ccccc|c} 1 & -2 & -1 & 3 & 1 & 5 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Rightmost column has no leading 1, so the system is consistent.
- The 2nd, 4th, 5th columns (left of the vertical bar) have no leading 1.

  So there are infinitely many solutions, and we need 3 parameters to describe the solutions.

b) (Source: Lecture 3a)

- If $M = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix}$,

  What is M^T ?

  $$M^T = \boxed{\begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}}$$

- What is the transpose of A^T ?

  $$(A^T)^T = \boxed{A}$$

Q5 Key

a. What is the size of the product

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \quad ?$$

$2\times\cancel{3}$ $\quad \cancel{3}\times 2$

Answer: $\boxed{2\times 2}$

b. What is the size of the product

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \quad ?$$

$3\times\cancel{2}$ $\quad \cancel{2}\times 3$

Answer: $\boxed{3\times 3}$

c. What is the size of the product

$$\begin{bmatrix} 1 & 2 & \pi & 5 \\ 3 & 0 & 0 & 2 \\ 0 & 9 & 1 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad ?$$

$5\times\cancel{4}$ $\quad \cancel{4}\times 1$

Answer: $\boxed{5\times 1}$

d. What is the size of the product

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & \pi & 5 \\ 3 & 0 & 0 & 2 \\ 0 & 9 & 1 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \quad ?$$

$2\times\cancel{5}$ $\quad \cancel{5}\times 4$

Answer: $\boxed{2\times 4}$

e. Simplify each expression into a single matrix, or state that it does not exist.

i. $$\left(\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}+\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\right)^T = \begin{bmatrix} 2+3 & 1-1 \\ 0+2 & -1+0 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 \\ 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & 2 \\ 0 & -1 \end{bmatrix}$$

ii. $$\begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}$$

iii. $$2\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 1 \\ -2 \end{bmatrix}$$

iv. $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Q6 Key

a. Write down a 2x2 matrix which commutes with every 2x2 matrix

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the identity matrix commutes with every 2x2 matrix

b. Write down another 2x2 matrix which commutes with every 2x2 matrix.

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ works

$$\begin{cases} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} \end{cases}$$

c. Is matrix multiplication commutative? No

d. Write down two (easy to remember) matrices which do not commute with each other.

- If A has size 3×2 and B has size 2×3, AB and BA are defined but they have different sizes, so $AB \neq BA$

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 8 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 4 & 0 \end{bmatrix}$$

Ref: Matrix multiplication (Lecture 4a, 4b)

Q7 Key

Similar to
Exercise 3
Lecture 5a

a. Suppose $A, B,$ and C are 2×2 matrices and $\det(C) = 5$.

Rewrite the matrix equation

$$AC - BC = 4CB$$

as a formula for A.

Answer

a. Since $\det(C) \neq 0$, C^{-1} exists.

$$AC - BC = 4CB$$
$$(A-B)C = 4CB$$
$$(A-B)CC^{-1} = 4CBC^{-1}$$
$$A - B = 4CBC^{-1}$$
$$A = 4CBC^{-1} + B$$

b. Sanity check

$4CBC^{-1} + B$

$$AC - BC \stackrel{?}{=} 4CB$$
$$(4CBC^{-1} + B)C - BC$$
$$4CB\underbrace{C^{-1}C}_{Id} + \underbrace{BC - BC}_{\vec{0}} = 4CB \checkmark$$

Q8 Key

Similar to Lecture 5b Exercise 9

a) $$M := \begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ is invertible

Why? • $\det(M) = 2 \cdot 5 = 10 \neq 0$, and we know

if the determinant of a matrix is nonzero then the inverse of the matrix exists

• Alternatively, the rank of M is 7 and its size is 7×7, so M is invertible according to the invertibility-and-rank theorem

b) $$C := \begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$ is not invertible.

Why? C is equivalent to the 7×7 matrix $$\begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

which has determinant 0 and rank smaller than 7.

Q 9 Key

Source: Lecture 7a Exercise 3

a. A possible step-by-step process

For cofactor method, the most convenient choice is either the 4th column or 4th row.
Here, I choose the 4th column

$$\det\begin{bmatrix} 1 & -1 & 7 & 0 & 1 \\ 0 & 2 & 6 & 0 & 1 \\ 7 & 5 & -6 & 2 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix} = \underbrace{0 \cdot C_{14}}_{0} + \underbrace{0 \cdot C_{24}}_{0} + 2 \cdot C_{34} + \underbrace{0 \cdot C_{44}}_{0} + \underbrace{0 \cdot C_{54}}_{0}$$

$$= 2 \cdot \underbrace{(-1)^{3+4}}_{-1} \det\begin{bmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 4 & 3 \end{bmatrix}$$

$R_4 \mapsto -R_1 + R_4$ does not change the determinant

$$= 2 \cdot (-1) \det\begin{bmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \mapsto -3R_3 + R_4$ does not change the determinant

$$= -2 \cdot \det\begin{bmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= -2 \quad 1 \cdot 2 \cdot -1 \cdot 2$$

$$= \boxed{8}$$

b. Yes, M is invertible because $\det(M) \neq 0$.

Con't below (for alternative solution)

Key (Continued) Source: Lecture 7a Exercise 3

a. A possible step-by-step process

For cofactor method, the most convenient choice is either the 4th column or 4th row.
Here, I choose the 4th row.

$$\det\begin{bmatrix} 1 & -1 & 7 & 0 & 1 \\ 0 & 2 & 6 & 0 & 1 \\ 7 & 5 & -6 & 2 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix} = a_{41}C_{41} + a_{42}C_{42} + a_{43}C_{43} + a_{44}C_{44} + a_{45}C_{45}$$

$$C_{41} = (-1)^{4+1}\det\begin{bmatrix} -1 & 7 & 0 & 1 \\ 2 & 6 & 0 & 1 \\ 5 & -6 & 2 & 4 \\ -1 & 4 & 0 & 3 \end{bmatrix}$$

the original matrix without the 4th row, 1st column

$$C_{42} = (-1)^{4+2}\det\begin{bmatrix} 1 & 7 & 0 & 1 \\ 0 & 6 & 0 & 1 \\ 7 & -6 & 2 & 4 \\ 1 & 4 & 0 & 3 \end{bmatrix}$$

the original matrix without the 4th row, 2nd column

I did not need to compute $C_{41}, C_{42}, C_{44}, C_{45}$ because $a_{41} = a_{42} = a_{44} = a_{45} = 0$, but I wanted to show you examples for computing cofactors

$$C_{43} = \underbrace{(-1)^{4+3}}_{-1}\det\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 7 & 5 & 2 & 4 \\ 1 & -1 & 0 & 3 \end{bmatrix}$$

the original matrix without the 4th row, 3rd column

Cofactor along the 3rd column

$$= -1 \cdot \left[0 + 0 + 2 \cdot (-1)^{3+3}\det\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix} + 0 \right]$$

$$= -1\left(2 \cdot \det\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\right) = -1 \cdot 2 \cdot \underbrace{(1 \cdot 2 \cdot 2)}_{\text{diagonal entries of an upper triangular matrix}}$$

$$= -8$$

$$\text{So, }\det\begin{bmatrix} 1 & -1 & 7 & 0 & 1 \\ 0 & 2 & 6 & 0 & 1 \\ 7 & 5 & -6 & 2 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix} = a_{41}C_{41} + a_{42}C_{42} + a_{43}C_{43} + a_{44}C_{44} + a_{45}C_{45}$$

$$= 0 \cdot C_{41} + 0 \cdot C_{42} + (-1)C_{43} + 0 \cdot C_{44} + 0 \cdot C_{45}$$

$$= 0 + 0 + (-1)(-8) + 0 + 0$$

$$= \boxed{8}$$

b. Yes, M is invertible because $\det(M) \neq 0$.

Q 10 Key

(Source: Lecture 7b Exercise 8)

a. Find all eigenvectors of $A := \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ with eigenvalue $\lambda = 0$

Set $A v = 0 v$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right]$$

$$R_1 \mapsto \tfrac{1}{2}R_1 \quad \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right]$$

$$R_2 \mapsto R_1 + R_2 \quad \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t$

$$2x - y = 0 \Rightarrow 2x - t = 0 \Rightarrow 2x = t \Rightarrow x = \tfrac{1}{2}t$$

The eigenvectors of A with eigenvalue $\lambda = 0$ are of the form

$$\boxed{\begin{pmatrix} \frac{1}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \text{ for nonzero } t.}$$

b. Check Compute $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix}$. Is it equal to $0\begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix}$?

c. If $M\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of M?

Yes. This means $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue 0

d. If $M\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean 0 is an eigenvalue of M? Yes.

e. If $M\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of M?

No. Eigenvector cannot be a zero matrix.

f. Suppose $B\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$. Write down one the eigenvalues of the matrix B.

- $\lambda = 2$ is an eigenvalue of B because $B\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Q11 Key

Lecture 7b, 8a

a. Suppose M is a 4×4 matrix.

Suppose $(M-5\,\mathrm{Id})\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ has a unique solution.
(exactly one)

- Is it enough information to determine whether 5 is an eigenvalue of M? (If so, state whether 5 is an eigenvalue.) yes.

  This means the only $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ satisfying $M\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=5\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ is $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$,

  so 5 is <u>not</u> an eigenvalue of M.

- Is it enough information to determine whether -5 is an eigenvalue of M? (If so, state whether -5 is an eigenvalue.)

  Not enough information about -5

b. Suppose $(M-5\,\mathrm{Id})\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ has infinitely many solutions.

- Is it enough information to determine whether 5 is an eigenvalue of M? (If so, state whether 5 is an eigenvalue.) yes.

  This means there are non-zero vectors $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ satisfying $M\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=5\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$

  so 5 is an eigenvalue of M.

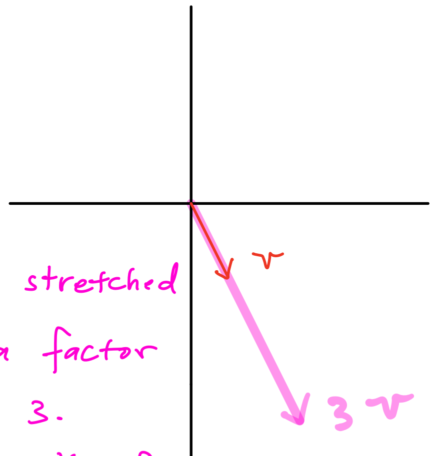
- Is it enough information to determine whether -5 is an eigenvalue of M? (If so, state whether -5 is an eigenvalue.)

  Not enough information about -5

Q 12 Key

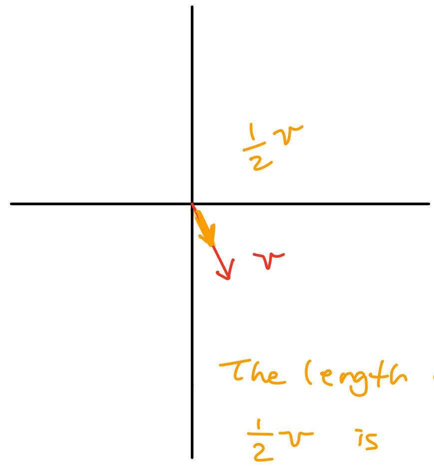
a. v is shown below.

Sketch $3v$ and $\frac{1}{2}v$.



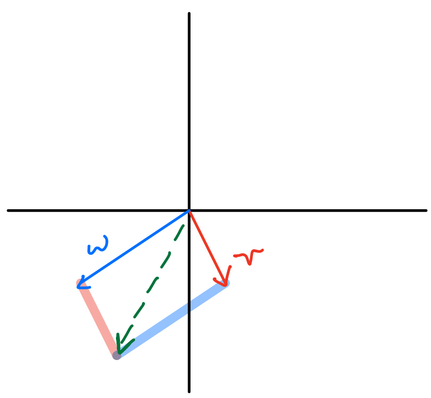
v is stretched by a factor of 3.

The length of $3v$ is three times $|v|$.



The length of $\frac{1}{2}v$ is half of $|v|$

b. sketch $v + w$, where v and w are shown below



$v + w$ are drawn in dashed line

Q 13 Key

$$A^4\vec{v} = A^3 A\vec{v}$$

$= A^3 \lambda \vec{v}$ by assumption $A\vec{v} = \lambda\vec{v}$

$= \lambda A^3 \vec{v}$

$= \lambda A^2 A\vec{v}$

$= \lambda A^2 \lambda \vec{v}$ by assumption $A\vec{v} = \lambda\vec{v}$

$= \lambda\lambda A^2 \vec{v}$

$= \lambda\lambda A A\vec{v}$

$= \lambda\lambda A \lambda\vec{v}$ by assumption $A\vec{v} = \lambda\vec{v}$

$= \lambda\lambda\lambda A\vec{v}$

$= \lambda\lambda\lambda\lambda \vec{v}$ by assumption $A\vec{v} = \lambda\vec{v}$

$= \lambda^4 \vec{v}$

We have shown that $A^4\vec{v} = \lambda^4\vec{v}$, so λ^4 is an eigenvalue of A^4 with $\vec{v}$ as a λ^4-eigenvector of A^4.

Q 14 Key

a. False. Counterexample: $2x+3y+4z=0$, $x+2y+2z=1$ has 3 variables.

The augmented matrix has 2 rows.

b. False. Counterexample: $x+y=0$, $x-y=0$ has exactly one solution

c. True. Applying a sequence of elementary operations results in a system which has the same set of solutions as the original system.

d. True. Applying a sequence of elementary operations results in a system which has the same set of solutions as the original system.

e. False. Counterexample: $M=\begin{bmatrix}1&1\\-1&0\end{bmatrix}$, $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $B=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Q 15 Key

a) True. $\det(AB)=\det(A)\det(B)$ is a property of det.

b) True. A square matrix is invertible iff its determinant is nonzero, so C^{-1} exists.

We know $\det(C)\det(C^{-1})=\det(CC^{-1})=\det(Id)=1$

So $-2\det(C^{-1})=1$.

c) False. Counterexample: Let $D=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

$$\det(-2D)=\det\left(\begin{bmatrix}-2&0\\0&-2\end{bmatrix}\right)=4$$

$$-2\det(D)=-2\det\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=-2$$

d) False. Counterexample: Let $E=F=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$\det(E+F)=\det\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)=4$$

$$\det(E)+\det(F)=1+1=2$$

Q16 Key

Answer whether each statement is TRUE or FALSE.

If true, give a justification. If false, provide a counterexample.

a) True. Ref: Lec 5a

Justification: A^{-1} exists, so $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

b) True. Ref: Lec 6a

Justification: A square matrix is invertible if and only if its determinant is nonzero.

c) True. Ref: Lec 5b

Justification: An $n \times n$ matrix is invertible if and only if its rank is n.

d) True. Ref: Lec 5b

Justification: The algorithm starting from $[A \mid Id]$ tells us that if we can perform row operations to get to $[Id \mid B]$ then B is the inverse of A.

e) True. Ref: Lec 5b

Justification: The algorithm starting from $[A \mid Id]$ tells us that if A has an inverse then we must be able to perform row operations to get to $[Id \mid B]$.

Q 17 <u>Key</u> Ref: Lec 8a

a. Compute the polynomial $P_A(x) = \det(x\,\mathrm{Id} - A)$

for $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$. What are the eigenvalues of A ?

$$\det(x\,\mathrm{Id}-A) = \det\left(\begin{bmatrix} x-1 & -4 & -5 \\ 0 & x+2 & -6 \\ 0 & 0 & x-3 \end{bmatrix}\right)$$

$$= (x-1)(x+2)(x-3)$$

because the matrix is upper triangular, simply take the product of the diagonal entries

The eigenvalues of A are 1, −2, 3.

b. Compute the polynomial $P_A(x) = \det(x\,\mathrm{Id} - A)$

for $A = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. What are the eigenvalues of A ?

$$\det(x\,\mathrm{Id}-A) = \det\left(\begin{bmatrix} x & -2 & 1 \\ 1 & x-3 & 0 \\ \boxed{0 \quad 0 \quad x-2} \end{bmatrix}\right)$$

I will use cofactor method along the 3rd row because there are many 0s

$$= (x-2)(-1)^{3+3} \det\begin{bmatrix} x & -2 \\ 1 & x-3 \end{bmatrix}$$

$$= (x-2)\quad (x(x-3) - (1)(-2))$$

$$= (x-2)(x^2 - 3x + 2)$$

$$= (x-2)(x-1)(x-2)$$

$$= (x-1)(x-2)^2$$

The eigenvalues of A are 1 and 2.