Math 3333 Linear Algebra

Exercises from Lectures 1 to 8 and Lecture 9a only

Summary of skills and concepts:

Perform row reduce step by step

Compute the rank of a matrix

Solving a linear system

- no solution
- exactly one solution
- colution set with 1 parameter, 2 parameters or more

Transpose of a matrix

scalar multiplication

matrix addition

matrix multiplication

Showing that matrix multiplication does not commute

Finding matrices which commute with other matrices of the same size

Inverse of a matrix

Rearranging equations

Determining whether a matrix is invertible

- using rank
- using determinant
- knowing that if the matrix is not square then it's not invertible

Computing determinant

- using row reduce and upper triangular matrix
- using cofactor expansion

Computing eigenvectors

- Given a number λ, find eigenvectors or determine it doesn't exist

Eigenvalues and eigenvectors

- Given the solution set of a matrix equation, determine whether a vector is an eigenvector & whether a number is an eigenvalue

Geometric meaning of vector arithmetic in 2D

- scalar multiplication
- vector addition

Dimension and shape of a solution set (see Q3)

Performing sanity checks after computing a solution

Q1

a. Use ① augmented matrix and ② row reduce to find all solutions to the linear system.

$$x + y + z = 3$$
$$x + y + 2z = 4$$
$$y + 2z = 2$$

b. * Write down a matrix multiplication which you can perform to verify your solution.

* Perform the matrix multiplication.

Q2 Let $M := \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix}$

a) There are many REF matrices equivalent to M.

Find an REF matrix equivalent to M.

Label all steps.

b) Use this REF matrix to find the rank of M.

c) Write down a 4×3 matrix with rank 2 or write "impossible".

Q 3

a. How many parameters are needed to describe all solutions to the (consistent) system

$$\begin{cases} a - 2b \quad + d = 2 \\ \qquad\quad c - 2d = 1 \end{cases} \quad ?$$

b. Describe all solutions to the (consistent) system

c. What is the dimension of this solution set ?

d. What is the shape of this solution set ?

Q 4

a) Consider a system of linear equations whose associated augmented matrix has an REF $\left[\begin{array}{ccccc|c} 1 & -2 & -1 & 3 & 1 & 5 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

(Do not look for the solutions.)

How many solutions does the original system have ?

If there are infinitely many solutions, how many parameters do you need to describe the solution set ?

b)
- If $M = [5 \quad 2 \quad 6]$, What is M^T ?
- What is the transpose of A^T ?

Q5 a. What is the size of the product

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \quad ?$$

b. What is the size of the product

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \quad ?$$

c. What is the size of the product

$$\begin{bmatrix} 1 & 2 & \pi & 5 \\ 3 & 0 & 0 & 2 \\ 0 & 9 & 1 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad ?$$

d. What is the size of the product

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & \pi & 5 \\ 3 & 0 & 0 & 2 \\ 0 & 9 & 1 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \quad ?$$

e. Simplify each expression into a single matrix, or state that it does not exist.

i. $\left(\begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\right)^T$

ii. $\begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}$

iii. $2\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

iv. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q6

a. Write down a 2x2 matrix which commutes with every 2x2 matrix

b. Write down another 2x2 matrix which commutes with every 2x2 matrix.

- How do you know it commutes with every 2x2 matrix?

c. Is matrix multiplication commutative?

d. Write down two (easy to remember) matrices which do not commute with each other.

Q7

a. Suppose A, B, and C are 2×2 matrices and $\det(C) = 5$.

Rewrite the matrix equation

$$AC - BC = 4CB$$

as a formula for A.

b. What should you do to verify your answer?

Q8

a) Let $M := \begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Without performing the $[M \mid Id] \rightsquigarrow [REF \mid \;]$ algorithm,

determine whether M is invertible

b) The same question for

$$C := \begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Hint: For part (a), M is an upper triangular matrix,
so we can quickly compute $\det(M)$.
Do you remember how? (Look at the diagonal entries)

Q9

a. Let $M := \begin{bmatrix} 1 & -1 & 7 & 0 & 1 \\ 0 & 2 & 6 & 0 & 1 \\ 7 & 5 & -6 & 2 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix}$

Compute $\det(M)$.

Explain each step of your process carefully.

b. Is M invertible?

Q10

a. Find all eigenvectors of $A := \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ with eigenvalue 0

b. How can you check this by hand?

c. If $M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of M?

d. If $M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean 0 is an eigenvalue of M?

e. If $M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of M?

f. Suppose $B \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

Write down one of the eigenvalues of the matrix B.

Q 11 Suppose M is a 4×4 matrix.

a. Suppose $(M - 5\,Id)\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has a unique solution.
(exactly one)

- Is it enough information to determine whether 5 is an eigenvalue of M?
  (If so, state whether 5 is an eigenvalue.)
  Explain.

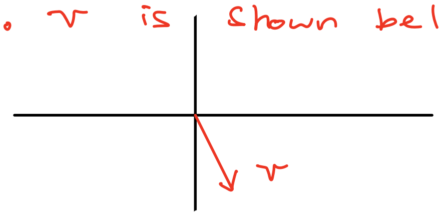
- Is it enough information to determine whether -5 is an eigenvalue of M?
  (If so, state whether -5 is an eigenvalue.)
  Explain.

b. Suppose $(M - 5\,Id)\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has infinitely many solutions.

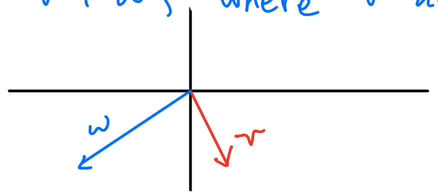
- Is it enough information to determine whether 5 is an eigenvalue of M?
  (If so, state whether 5 is an eigenvalue.)
  Explain.

- Is it enough information to determine whether -5 is an eigenvalue of M?
  (If so, state whether -5 is an eigenvalue.)
  Explain.

Q12 a. v is shown below. Sketch $3v$ and $\frac{1}{2}v$.



b. sketch $v+w$, where v and w are shown below



Q13

Suppose A has an eigenvalue λ with $\vec{v}$ as a λ-eigenvector. (That is, suppose we know $A\vec{v} = \lambda\vec{v}$.)

Show that λ^4 is an eigenvalue of A^4.

(That is, show that $A^4\vec{w} = \lambda^4\vec{w}$ for some vector $\vec{w}$.)

Q14 Answer whether each statement is TRUE or FALSE.

If true, give a justification. If false, provide a counterexample.

a. If a linear system has n variables and m equations, then the augmented matrix has n rows.

b. If a linear system is consistent, it must have infinitely many solutions.

c. If an elementary operation is done to a consistent linear system, the new system must be consistent.

d. If a sequence of elementary operations results in an inconsistent system, the original system is inconsistent.

e. If A and B are two REF matrices which are equivalent to a matrix M, then $A = B$.

Q15 Answer whether each statement is TRUE or FALSE.

If true, give a justification. If false, provide a counterexample.

All matrices below are $n \times n$ matrices.

a) $\det(AB) = \det(A)\det(B)$

b) If $\det(C) = -2$, then C^{-1} exists and $\det(C^{-1}) = \frac{-1}{2}$.

c) $\det(-2D) = -2\det(D)$

d) $\det(E+F) = \det(E) + \det(F)$

Q16 Answer whether each statement is TRUE or FALSE.

If true, give a justification. If false, provide a counterexample.

a.) If A is an invertible 4×4 matrix, then the equation $A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ is consistent.

b.) If A is a 4×4 matrix and $\det(A) = -2$, then A^{-1} exists.

c.) If A is a 4×4 matrix and $\operatorname{rank}(A) = 4$, then A^{-1} exists.

d.) If A can be turned into the identity matrix Id by applying a sequence of row operations, then A must be invertible.

e.) If A is invertible, then A can be turned into the identity matrix Id by applying a sequence of row operations

Q 17

Ref: Lec 8a

a. Compute the polynomial $P_A(x) = \det(x\,\mathrm{Id} - A)$

for $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$. What are the eigenvalues of A?

b. Compute the polynomial $P_A(x) = \det(x\,\mathrm{Id} - A)$

for $A = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. What are the eigenvalues of A?