Contents

1 Question (lecture11a.pdf)

$$\text{Let} \quad M := \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 1 & 0 & 2 & -1 & 6 \\ 2 & 4 & 0 & 6 & 0 \end{bmatrix}.$$

Is $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ in $\ker(M)$?

Solution: We can compute

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 0 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So the answer is $\boxed{Yes}$.

2 Question (lecture12a.pdf)

Let

$$B := \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}.$$

Find a spanning set for the kernel of B.

Solution: First, we compute the kernel of B, that is, find *all* solutions to

$$B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Perform row reduce on the augmented matrix associated to $B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 2 & -4 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the 2nd column has no leading 1, let $y = t$. The 2rd column also has no leading 1, so let $z = r$. Then $x = 2y + z = 2t + r$, so

$$\ker B = \left\{ \begin{bmatrix} 2t + r \\ t \\ r \end{bmatrix} \text{ for } t, r \text{ in } \mathbb{R} \right\} = \left\{ t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ where } t, r \text{ are in } \mathbb{R} \right\}$$

So a possible spanning set for the kernel of B is

$$\boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}}.$$

3 Question (lecture14a.pdf)

Find a basis for the kernel of

$$B := \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}$$

Solution: The spanning set given in the solution key to the previous question works as a basis because the algorithm given in the solution key is Algorithm 1 ("Basis algorithm for the kernel of a matrix") in lecture14a.pdf. So a possible basis for the kernel of B is the set

$$\boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}}.$$

4 Question (lecture12a.pdf, lecture12b.pdf)

(For a similar problem, see "SAMPLE STUDENT PROOF" for Exercise 3 of lecture12a.pdf.)

Let $V = \left\{ \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} \text{ where } a, b \text{ are real numbers} \right\}$, that is, the subspace of $\mathbb{R}^3$ consisting of height-3 vectors whose 3rd entry is the sum of the first two entries.

Let

$$S := \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

a.) Show that S is a spanning set for V.

Solution: First, we verify that the vectors of S are in V: $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in V because $1+2=3$, and $\begin{bmatrix}0\\1\\1\end{bmatrix}$ is in V because $0+1=1$.

Next, we show that every vector in V can be written as a linear combination of the vectors of S:
Let v be in V. That is, $v = \begin{bmatrix}a\\b\\a+b\end{bmatrix}$ for some a, b in $\mathbb{R}$. We need to show that the equation

$$x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}a\\b\\a+b\end{bmatrix}$$

has at least one solution. In other words, we need to show that

$$\begin{bmatrix}1&0\\2&1\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}a\\b\\a+b\end{bmatrix} \tag{1}$$

has at least one solution.
Turn (1) into an augmented matrix, then row reduce.

$$\left[\begin{array}{cc|c}1&0&a\\2&1&b\\3&1&a+b\end{array}\right] \to \left[\begin{array}{cc|c}1&0&a\\0&1&b-2a\\3&1&a+b\end{array}\right] \to \left[\begin{array}{cc|c}1&0&a\\0&1&b-2a\\0&1&a+b-3a\end{array}\right] = \left[\begin{array}{cc|c}1&0&a\\0&1&b-2a\\0&1&b-2a\end{array}\right] \to \left[\begin{array}{cc|c}1&0&a\\0&1&b-2a\\0&0&0\end{array}\right]$$

Since there is no leading 1 in the right-most column of an equivalent REF matrix, the equation (1) is consistent. So v can be written as a linear combination of the two vectors of S. This means the set S spans V.

b.) Determine whether S is linearly independent or linearly dependent. Explain.

Solution:
We need to check whether that the equation

$$x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

has a non-trivial solution. In other words, we need to check whether

$$\begin{bmatrix}1&0\\2&1\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{2}$$

has a solution other than $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$
Turn (2) into an augmented matrix, then row reduce.

$$\left[\begin{array}{cc|c}1&0&0\\2&1&0\\3&1&0\end{array}\right] \to \left[\begin{array}{cc|c}1&0&0\\0&1&0\\3&1&0\end{array}\right] \to \left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&1&0\end{array}\right] = \left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&1&0\end{array}\right] \to \left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\end{array}\right]$$

Every column (to the left of the vertical line) of an equivalent REF matrix has a leading 1, so (2) has one unique solution (the trivial solution). Therefore, the only way to write the zero vector as a linear combination of the vectors in S is

$$0\begin{bmatrix}1\\2\\3\end{bmatrix} + 0\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Thus, the set S is $\boxed{\text{linearly independent}}$.

c.) Is S is a basis for V? Explain.

Solution: $\boxed{\text{Yes, } S \text{ is a basis for } V.}$ Part (a) tells us that S is a spanning set for V and part (b) tells us that S is linearly independent, so S is a basis for V (per definition of basis).

5 Question (lecture12a.pdf, lecture12b.pdf)

Let $V = \left\{ \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} \text{ where } a, b \text{ are real numbers} \right\}$ the subspace of $\mathbb{R}^3$ consisting of height-3 vectors whose 3rd entry is the sum of the first two entries.

Find a spanning set for V that is different from the spanning set given in the previous question.

Solution: For any numbers a, b, we can write $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

So the set of vectors $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}}$ works as a spanning set for V (that is different than the spanning set given in the previous question).

6 Question (lecture12a.pdf)

$$\text{Let } A := \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \end{bmatrix}.$$

a. Compute $\ker(A)$, i.e., find all solutions to $A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Parametrize the solution set *using parameters s and t.*

Solution: Perform row reduce on the augmented matrix associated to $A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & 3 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Since the 2nd and 4th columns have no leading one (while the 1st and 3rd columns do), let $\boxed{b = s}$ and $\boxed{d = t}$. Then

$$\begin{aligned} a + 2b + c + 2d &= 0, \\ c + d &= 0 \end{aligned}$$

so $\boxed{c = -t}$ and $a + 2s + (-t) + 2t = 0$, that is, $\boxed{a = -2s - t}$.

$$\boxed{\ker B = \left\{ \begin{bmatrix} -2s - t \\ s \\ -t \\ t \end{bmatrix} \text{ for } s, t \text{ in } \mathbb{R} \right\} = \left\{ s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \text{ where } s, t \text{ are in } \mathbb{R} \right\}}$$

(Recommended sanity check: Pick a nonzero vector $\mathbf{v}$ in your solution set, compute $A\mathbf{v}$, and verify that it is equal to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

b. Find two vectors $\mathbf{v}$ and $\mathbf{w}$ where you can rewrite your parametrization of the kernel of A as $s\mathbf{v} + t\mathbf{w}$.

Solution: A possible answer: $\mathbf{v} := \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{w} := \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ (there are many possible answers)

c. Finish each of the following sentences.

(i) The kernel of A is equal to *the set of all linear combinations* of the vectors ____________________

Solution: A possible answer: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ (there are many possible answers)

(ii) The kernel of A is equal to the *span* of the vectors ____________________

Solution: A possible answer: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ (there are many possible answers)

(iii) The kernel of A is equal to the *image* of a matrix ____________________

Solution:
Recall Fact 1 of lecture 12a says: im M = span of the column vectors of M, e.g. Exercise 2:

$$\text{im} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \text{span of the columns.}$$

Answer: $\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$ (the concatenation of the two vectors I have computed above. If you choose two different vectors, your concatenation would be different.)

7 Question (lecture13a.pdf)

a.) Determine whether $S := \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$ is or is **not** a basis of $\mathbb{R}^3$.

b.) Use computation (show all steps of the computation) and words (write in complete sentences).

Solution: a.) Answer: S is not a basis for $\mathbb{R}^3$.

b.) There are many ways to show that S is not a basis of $\mathbb{R}^3$.

Possible answer A: One way is to find a specific vector in $\mathbb{R}^3$ which cannot be written as a linear combination of the vectors in S. For example, you can show that the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ cannot be written as a linear combination of the vectors in S: Set up the augmented matrix with columns of S of the left and $\begin{bmatrix}1\\0\\0\end{bmatrix}$ on the right. After row reduce, you conclude that the system is inconsistent.

Possible Answer B: We can show S is not linearly independent. Set the zero vector as a linear combination of the three vectors in S. The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 6 & 3 & 0 \\ 2 & 7 & 1 & 0 \\ 1 & 4 & 1 & 0 \end{array}\right].$$

After row reduce, you should get an REF matrix with all 0s on the third row (there are many possibly REF matrices):

$$\left[\begin{array}{ccc|c} 1 & 6 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This means we can write the zero vector as a linear combination of the vectors of S in more than one way. So S is not linearly independent.

8 Question (lecture13a.pdf)

a.) Determine whether the set of vectors $S := \left\{ \begin{bmatrix}5\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix} \right\}$ is or is *not* a basis for $\mathbb{R}^3$.

b.) Use computation (show all steps of the computation) and words (write in complete sentences).

Solution: a.) Answer: S is a basis for $\mathbb{R}^3$.

b.) Let A be the concatenation of these vectors:

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

To determine whether it is a basis or not, we can **either** solve the equation $A\mathbf{v} = \mathbf{b}$ for an arbitrary vector $\mathbf{b} \in \mathbb{R}^3$, **or** since the subspace is $\mathbb{R}^3$, we can just find the rank of A.

Let us do the former: Let $\mathbf{b} \in \mathbb{R}^3$ be an arbitrary vector. Then to solve the equation $A\mathbf{v} = \mathbf{b}$, we consider the associated augmented matrix $[A \mid \mathbf{b}]$:

$$\left[\begin{array}{ccc|c} 5 & 1 & 3 & b_1 \\ 2 & 0 & -1 & b_2 \\ -1 & 1 & 0 & b_3 \end{array}\right]$$

Let us use row operations to simplify this system:

$$\left[\begin{array}{ccc|c} 5 & 1 & 3 & b_1 \\ 2 & 0 & -1 & b_2 \\ -1 & 1 & 0 & b_3 \end{array}\right] \longmapsto \left[\begin{array}{ccc|c} 0 & 6 & 3 & b_1 + 5b_3 \\ 0 & 2 & -1 & b_2 + 2b_3 \\ -1 & 1 & 0 & b_3 \end{array}\right] \longmapsto \left[\begin{array}{ccc|c} 0 & 0 & 6 & b_1 - 3b_2 - b_3 \\ 0 & 2 & -1 & b_2 + 2b_3 \\ -1 & 1 & 0 & b_3 \end{array}\right]$$

where in the first step, the row operations are adding 2(Row 3) to Row 2 and 5(Row 3) to Row 1; and in the second step, the row operation is adding -3(Row 2) to Row 1. The corresponding system of equations to the matrix on the right is:

$$\begin{aligned} 6z &= b_1 - 3b_2 - b_3 \\ 2y - z &= b_2 + 2b_3 \\ -x + y &= b_3 \end{aligned}$$

From the first equation, we get that $z = \frac{b_1-3b_2-b_3}{6}$. Putting that in the second equation and solving gives us $y = \frac{1}{2}\left(b_2 + 2b_3 + \frac{b_1-3b_2-b_3}{6}\right)$. Putting these in the third equation and solving gives us $x = \frac{1}{2}\left(b_2 + 2b_3 + \frac{b_1-3b_2-b_3}{6}\right) - b_3$. Therefore, for every $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$, the system $A\mathbf{v} = \mathbf{b}$ has a unique solution. Therefore, the set of vectors S is indeed a basis of $\mathbb{R}^3$.

9 Question (lecture13b.pdf)

Write down a 4×5 matrix M such that the dimension of $\mathrm{im}(M)$ is 3.

Solution: The dimension of the image of a matrix is equal to the rank of the matrix. So write down any matrix M with 4 rows and 5 columns whose rank is 3. (Recall that rank of M is the same as the number of leading 1s in an REF matrix equivalent to M).

A possible matrix is $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}$

10 Question (lecture13b.pdf)

$$\text{Let } \; M := \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 1 & 0 & 2 & -1 & 6 \\ 2 & 4 & 0 & 6 & 0 \end{bmatrix}.$$

a.) What is the dimension of $\mathrm{im}(M)$?

Solution: I will follow "Algorithm 5, rephrased" in lecture13b.pdf:
I perform a few elementary row operations until I reach a row-echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 1 & 0 & 2 & -1 & 6 \\ 2 & 4 & 0 & 6 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

An REF matrix equivalent to M has 3 leading ones, so the dimension of $\mathrm{im}(M)$ is $\boxed{3}$.

b.) Now find a basis of $\mathrm{im}(M)$.

Solution: We will use the REF matrix equivalent to M (which has been computed in the previous solution box):
A possible basis for $\mathrm{im}(M)$ is the original columns of M corresponding to the columns in the REF matrix which have leading 1s, the 1st, 2nd, and 5th column: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \right\}$

11 Question (lecture14a.pdf)

Let B be a 4×5 matrix. After row reducing to an REF matrix, we learn that this REF matrix has 3 leading 1s.

a. What is the rank of B?

> **Solution:** The rank of B is 3 because the question says that an REF matrix equivalent to B has three leading 1s.

b. How many parameters are required to parametrize the kernel of B?

> **Solution:** Since B has 5 columns and an REF matrix corresponding to B has 3 leading 1s, we need $5-3=2$ parameters.

c. How many vectors are required to span the kernel of B?

> **Solution:** Also $5-3=2$.

12 Question (lecture14a.pdf)

Write down a 4×5 matrix M such that the dimension of $\ker(M)$ is 2.

> **Solution:** The dimension of the kernel of a matrix is equal to (the number of columns of the matrix) minus (the rank of the matrix). So write down any matrix M with 4 rows and 5 columns whose rank is 3.
>
> A possible matrix is $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}$