Lecture 9a

Vector Geometry

So far, we've focused on algebraic computations.
From now on, we shift focus to concepts and intuition.

Lecture 9 and 10

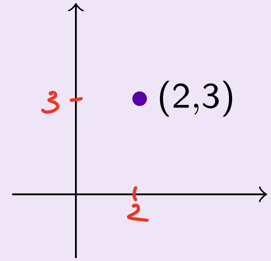
Visualizing vectors of height 2 and 3 geometrically, and translating ideas from the class so far into geometry.

Interpreting 2-vectors geometrically

We can visualize 2-vectors in the plane in two ways.

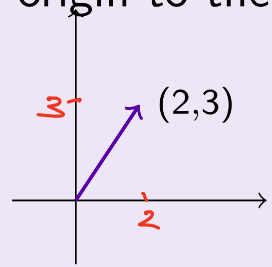
- We can interpret the entries as coordinates of a point.

$$\begin{bmatrix}2\\3\end{bmatrix} \xrightarrow{\text{...becomes...}}$$



- We can draw an arrow from the origin to the above point.

$$\begin{bmatrix}2\\3\end{bmatrix} \xrightarrow{\text{...becomes...}}$$



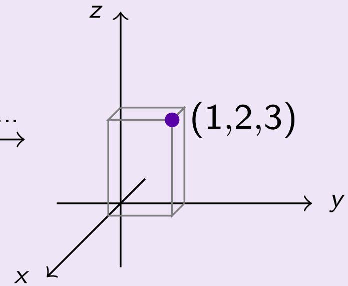
Confusingly, this arrow is often called a **(geometric) vector**.

Interpreting 3-vectors geometrically

We can do the same thing for 3-vectors.

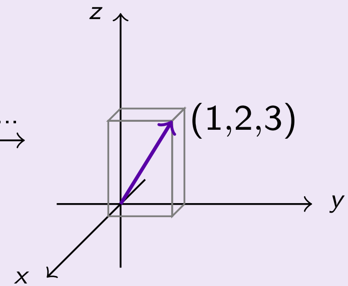
- We can interpret the entries as coordinates of a point.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \xrightarrow{\text{...becomes...}}$$



- We can draw an arrow from the origin to the above point.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \xrightarrow{\text{...becomes...}}$$



There is no standard for which variable corresponds to each axis.

Interpreting bigger vectors geometrically?

As 3D people, you can't visualize 4-dimensional space or higher directly. So we can't visualize larger vectors geometrically.

That said, we can run the intuition the other way, and use our algebraic knowledge of vectors to understand higher dimensional space.

Exercise 1(a)

Draw all solutions to the following system of linear equations.

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \\ y &= 1 \end{aligned}$$

Exercise 1(b)

Draw all solutions to the following system of linear equations.

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \\ x + y &= 1 \end{aligned}$$

Exercise 1(c)

Draw all solutions to the following system of linear equations.

$$\begin{aligned} x + y &= 3 \\ -x - y &= -3 \\ 2x + 2y &= 6 \end{aligned}$$

1a

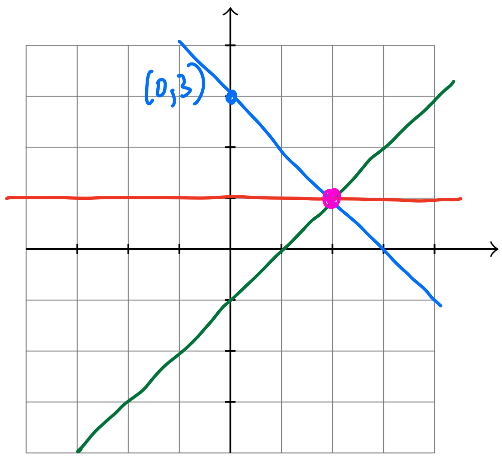
slope is -1

$x + y = 3 \iff y = -x + 3$

$x - y = 1 \iff y = x - 1$

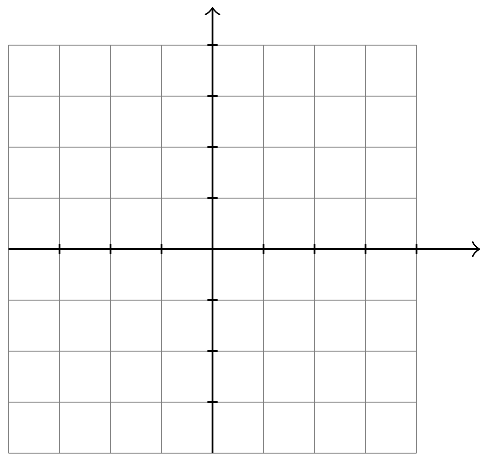
slope is 1

$y = 1$

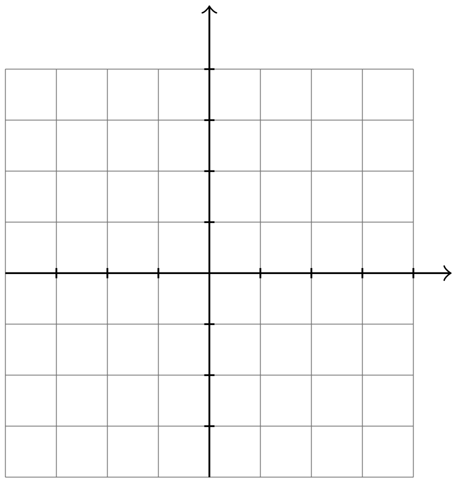


This system has exactly one solution, $(x,y) = (2,1)$

$$x + y = 3$$
$$x - y = 1$$
$$x + y = 1$$



$$x + y = 3$$
$$-x - y = -3$$
$$2x + 2y = 6$$



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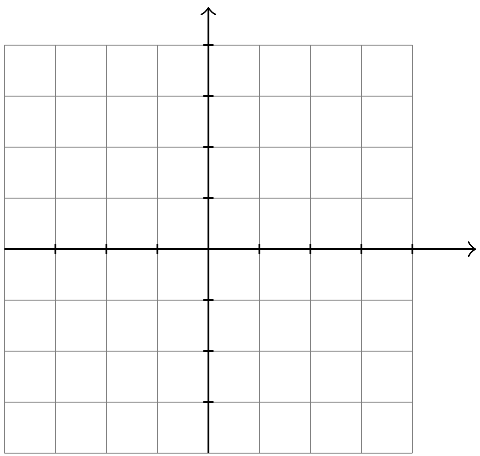
$$x + y = 3$$
$$x - y = 1$$
$$y = 1$$



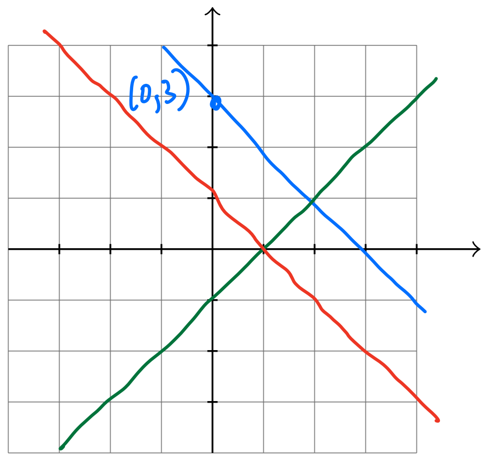
slope is -1

$$x + y = 3 \Leftrightarrow y = -x + 3$$
$$x - y = 1 \Leftrightarrow y = x - 1$$

slope is 1

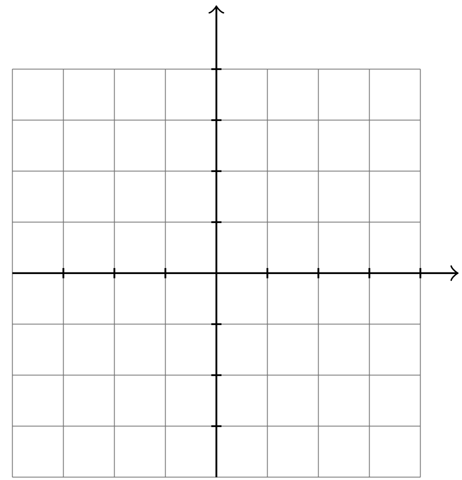
$$x + y = 1 \Leftrightarrow y = -x + 1$$

slope is -1

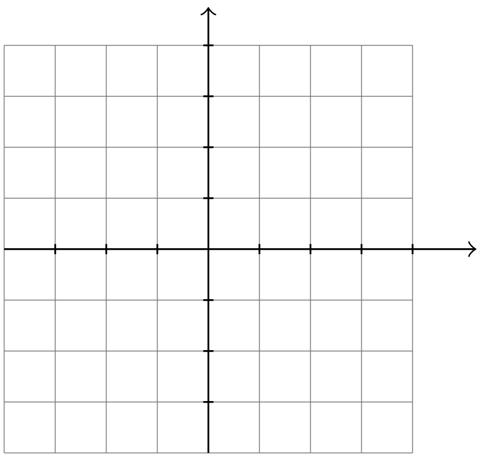


No points are solutions to this system

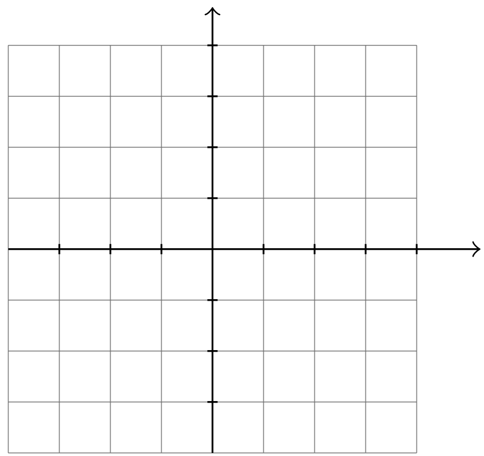
$$x + y = 3$$
$$-x - y = -3$$
$$2x + 2y = 6$$



$$\begin{aligned} x+y&=3\\ x-y&=1\\ y&=1 \end{aligned}$$



$$\begin{aligned} x+y&=3\\ x-y&=1\\ x+y&=1 \end{aligned}$$

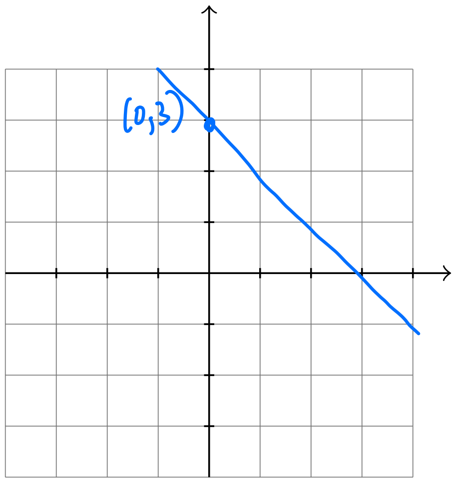


slope is -1

$y=-x+3 \iff x+y=3$

$y=-x+3 \iff -x-y=-3$

$y=-x+3 \iff 2x+2y=6$



The solutions are points $(x,y)=(t,-t+3)$ where t is a number

c

Now is a good time to introduce the idea of sets.

Sets

A **set** is a collection of objects, called **elements**...which are typically mathematical objects like numbers, vectors, or points.

Examples of sets

- The set of all cats in Oklahoma — a finite set
- The set of all even integers — an infinite set
- The set of solutions to a system of linear equations
  - can have 0 elts
  - can have exactly one element
  - can have infinitely many elements
- The set of eigenvalues of a matrix
- The set of all 2-vectors — an infinite set

always a finite set
(an nxn matrix can have at most n eigenvalues)

Defining sets

Sets can be defined by listing their elements...

$\{\text{cat } 1, \text{cat } 2, \ldots, \text{cat } d\}$, $\{-1, 2, 7\}$, $\{\ldots, -4, -2, 0, 2, 4, 6, 8, \ldots\}$

...parametrizing their elements...

$$\left\{ \begin{bmatrix} 1-2t \\ t \end{bmatrix} \text{ for all numbers } t \right\}, \quad \{ 2k \text{ for all integers } k \}$$

...or characterized by a property.

the set of solutions to $x + 2y + 3y = 6$, the set of integers divisible by 2

No elements?!

A set with no elements is called the empty set.

Graphing sets of vectors

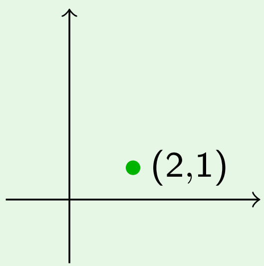
We can visualize a set of 2-vectors or 3-vectors by drawing the corresponding points in the plane or space, respectively.

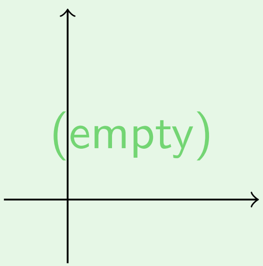
Example

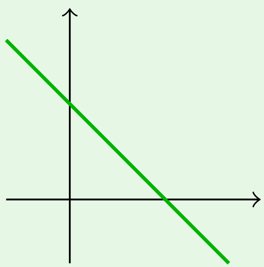
system of linear equations (SLE)

$x+y=3$	$x+y=3$	$x+y=3$
$x-y=1$	$x-y=1$	$-x-y=-3$
$y=1$	$x+y=1$	$2x+2y=6$

Solution set







What kinds of shapes can we get as the set of solutions to an SLE?
Let's start with one linear equation.

The shape of solutions to a linear equation

- The set of solutions to a linear equation in 2-variables

  $$ax + by = c \quad , \text{ e.g. } x - y = 0 \text{ or } 2x + 5y = -\pi$$

  is a line in the plane.
- The set of solutions to a linear equation in 3-variables

  $$ax + by + cz = d \quad , \text{ e.g. } x + y + z = 1 \text{ or } x - 5y - z = \frac{1}{7}$$

  is a plane in space.

We don't have the vocabulary for cases in higher dimension.

One exception

Technically, $0 = 1$ is a linear equation whose solutions are empty.

The shape of solutions to a system of linear equations

The solution set to a system of linear equations has a fixed shape that depends on the number of solutions and parameters.

# of solutions	Shape
No solutions	No points (empty)
One solution	One point
One parameter	A line
Two parameters	A plane
m parameters	?

→ dimension of the solution set is 1

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Let $y=t$

$x-y=0 \Rightarrow x=t$

$z=2$

The solution set is

$$\left\{ \begin{bmatrix} t \\ t \\ 2 \end{bmatrix} \text{ for a number } t \right\}$$

or $\left\{ t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \text{ for a number } t \right\}$

For this reason, the number of parameters needed is called the dimension of the solution set.

The shape of solutions to a system of linear equations

The solution set to a system of linear equations has a fixed shape that depends on the number of solutions and parameters.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $y = t$
$z = s$

$x - y + z = 1 \Rightarrow$
$x - t + s = 1 \Rightarrow$
$x = 1 + t - s$

The solution set is

$$\left\{ \begin{bmatrix} 1+t-s \\ t \\ s \end{bmatrix} \text{ for all numbers } t, s \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ for all numbers } t, s \right\}$$

# of solutions	Shape
No solutions	No points (empty)
One solution	One point → dimension of the solution set is 0
One parameter	A line → dimension of the solution set is 1
Two parameters	A plane → dimension of the solution set is 2
m parameters	?

For this reason, the number of parameters needed is called the dimension of the solution set.

We will often denote a set by a letter, for example:

Let L be the set of solutions to $2x - 3y = 3$.

Then $L = \left\{ \begin{bmatrix} \frac{3}{2} + \frac{3}{2}t \\ t \end{bmatrix} \text{ for all numbers } t \right\}$

$= \left\{ \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \text{ for all numbers } t \right\}$

A few sets come up so often that they have standard names.

Standardized set names

- $\mathbb{R}$ is the set of (real) numbers.
- $\mathbb{R}^2$ is the set of 2-vectors, which we can visualize as the plane.
- $\mathbb{R}^3$ is the set of 3-vectors, which we can visualize as space.
- For any n, $\mathbb{R}^n$ is the set of n-vectors.
- For any m, n, $\mathbb{R}^{m \times n}$ is the set of $m \times n$-matrices.

We can extend our dictionary between algebra and geometry.

n-dimensional geometry

Many of the formulas from 2D and 3D geometry extend to vectors of all sizes.

Geometry $\to$ algebra

- The length of a vector $\mathrm{v} := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^\top$ is
$$|\mathrm{v}| := \sqrt{v_1^2 + v_2^2 + \cdots v_n^2} = \sqrt{\mathrm{v} \cdot \mathrm{v}}$$
- The angle between two vectors v and w can be defined by
$$\cos(\text{the angle between v and w}) = \frac{\mathrm{v} \cdot \mathrm{w}}{|\mathrm{v}||\mathrm{w}|}$$

Example

The **length** of the four-dimensional vector $v := \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ is

$$|v| = \sqrt{v \cdot v} =$$

Compute the **angle** between v and v.

Answer:

$$\begin{aligned} \cos(\boxed{}) &= \frac{v \cdot v}{|v||v|} \\ &= \end{aligned}$$

Example

The **length** of the four-dimensional vector $v := \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ is

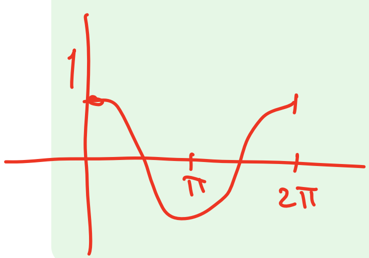
$$|v| = \sqrt{v \cdot v} = \sqrt{1^2 + 2^2 + 1^2 + 1^2} = \sqrt{7}$$

Compute the **angle** between v and v.

Answer:

$$\cos(\boxed{}) = \frac{v \cdot v}{|v||v|}$$

$$= \frac{1^2 + 2^2 + 1^2 + 1^2}{\sqrt{7}\sqrt{7}} = \frac{7}{7} = 1$$



$\cos(\boxed{?}) = 1$

$\therefore$ The angle between v and v is $\boxed{0.}$

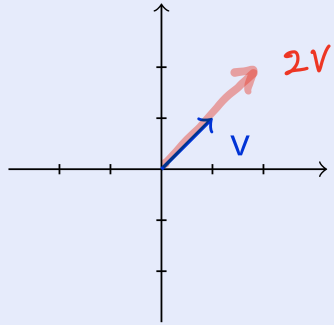
Dually, we can try to move algebraic ideas into geometry.

From algebra to geometry

- Multiplying v by a scalar c stretches c by a factor of c.

If $c := 2$

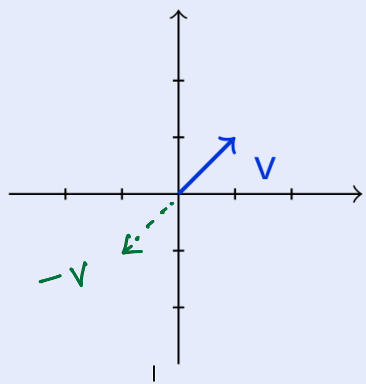
and $v := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



- Multiplying v by a scalar c stretches c by a factor of c.

If $c := -1$

$v := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

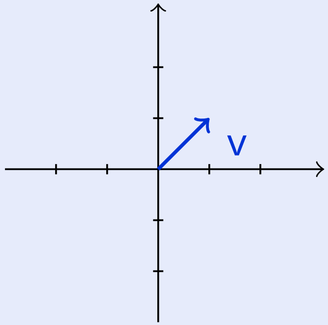


Dually, we can try to move algebraic ideas into geometry.

From algebra to geometry

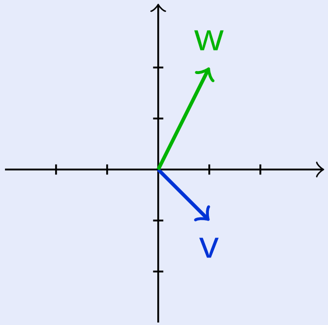
- Multiplying v by a scalar c stretches v by a factor of c.

$cv = ?$



- Adding v and w gives the new vector obtained by sliding the tail of one vector to the tip of the other.

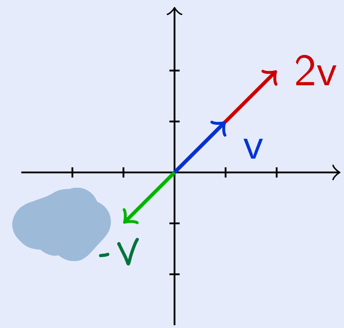
$v + w = ?$



Dually, we can try to move algebraic ideas into geometry.

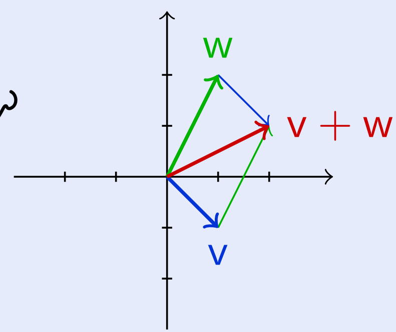
From algebra to geometry

- Multiplying v by a scalar c stretches c by a factor of c.



- Adding v and w gives the new vector obtained by sliding the tail of one vector to the tip of the other.

if v is not a scalar multiple of w



- Today: Vector geometry
- Next time: Matrix algebra to geometry