Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00	00	0000

# Lecture 4b

# Matrix Multiplication (properties of matrix arithmetic)

Droport	ioc of mo	trix arithmetic		
Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00		0000

## Last Time

Matrix multiplication violates one of the basic rules of traditional arithmetic: matrix multiplication does not always commute.

### Goal

What familiar properties does matrix arithmetic have?

Matrix	addition i	s associative and com	mutative	
Distributivity 0	Associativity 000	Transpose reverses multiplication order	Solving a matrix equation	Warning 0000

# Associativity of addition

$$A + (B + C) = (A + B) + C$$

So, we can write A + B + C to mean

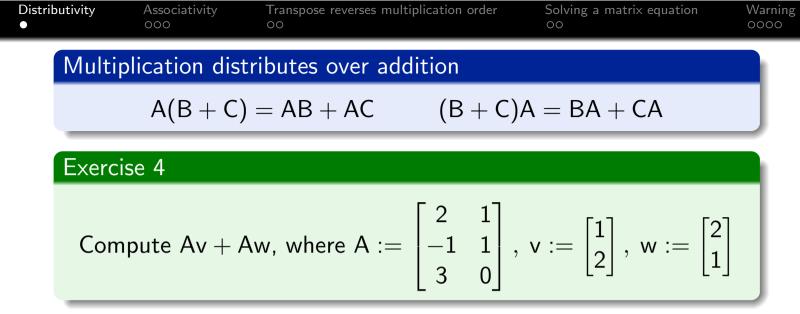
"A + (B + C) or (A + B) + C, your choice"

# Commutativity of addition

$$\mathsf{A} + \mathsf{B} = \mathsf{B} + \mathsf{A}$$

As a result, we can rearrange sums of matrices however we want.

$$A + B + C = C + A + B$$
$$= B + C + A$$
$$= C + B + A$$



Note: It takes less effort to compute A(v + w) than to compute Av + Aw.

Distributivity
 Associativity
 Transpose reverses multiplication order
 Solving a matrix equation
 Warning

 Multiplication distributes over addition
 
$$A(B + C) = AB + AC$$
 $(B + C)A = BA + CA$ 

 Exercise 4
 Compute Av + Aw, where  $A := \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$ ,  $v := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $w := \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ 

Note: It takes less effort to compute A(v + w) than to compute Av + Aw.

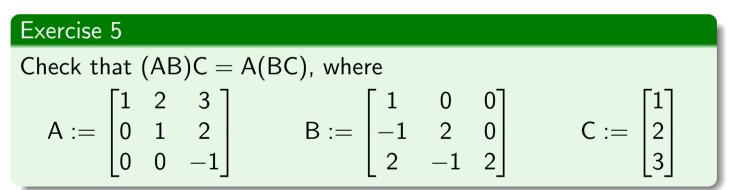
$$v + w = \begin{bmatrix} 1\\2 \end{bmatrix} + \begin{bmatrix} 2\\1 \end{bmatrix} = \begin{bmatrix} 3\\3 \end{bmatrix}$$
$$A(v + w) = \begin{bmatrix} 2 & 1\\-1 & 1\\3 & 0 \end{bmatrix} \begin{bmatrix} 3\\2 \end{bmatrix} = \begin{bmatrix} 3 \times 1\\23 + 1 \end{bmatrix} = \begin{bmatrix} 7\\0\\23 + 1 \end{bmatrix}$$

0	•00	00	00	0000		
Matrix multiplication is associative						

# Associativity of multiplication

$$A(BC) = (AB)C$$

So, we can write ABC to mean "A(BC) or (AB)C, your choice".



Compute A(BC) and (AB)C. Which took more computation?

$$Associativity 
o o 
Married 
O 
Compute A(BC) and (AB)C. Which took more computation?
$$BC = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.1 + 0.2 + 0.3 \\ 1.1 + 2.2 + 0.3 \\ 2.1 + -1.2 + 2.3 \\ 2.1 + -1.2 + 2.3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 3 & 0 & 4 \\ -2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 3 & 0 & 4 \\ -2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 3 & 0 & 4 \\ -2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 25 \\ 15 \\ -6 \end{bmatrix}$$$$

Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00	00	0000

## So far

- Matrix addition is both associative and commutative
- Matrix multiplication is associative but is not commutative

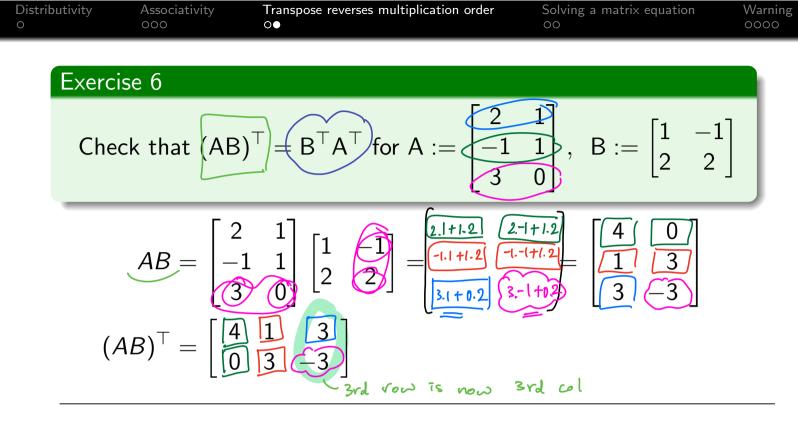
Can you think of a familiar arithmetic operation (from elementary school) which is not associative?

Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	•0	00	0000

# Transpose reverses the order of multiplication

$$(AB)^{\top} = B^{\top}A^{\top}$$

# Exercise 6 Check that $(AB)^{\top} = B^{\top}A^{\top}$ for $A := \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$ , $B := \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$



$$B^{\top} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \ A^{\top} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$
$$B^{\top} A^{\top} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{(1,1)}{(1,2+2,1)} & \frac{(1,2)}{(1,2+2,1)} \\ \frac{(2,1)}{(1,2+2,1)} & \frac{(2,1)}{(1,2+2,1)} \\ \frac{(2,1)}{(1,2+2,1)} & \frac{(2,1)}{(1,2+2,1)} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00	• O	0000

#### Theorem

Suppose A and B are matrices.

(
$$A^{T}$$
)<sup>T</sup> = A.  
( $rA$ )<sup>T</sup> =  $r(A^{T})$  if r is a number.  
( $A + B$ )<sup>T</sup> =  $A^{T} + B^{T}$ .

# Algebra with matrices

Using these rules, we can rearrange expressions involving matrices.

## Exercise 7

Solve the following expression for B. Write B in terms of A.  $(A^{\top} + 2B^{\top})^{\top} = 2A$ 

Distributivity 0	Associativity 000	Transpose reverses multiplication order	Solving a matrix equation ○●	Warning 0000
Theo	orem			
0	$(A^T)^T = A.$			
2	$(rA)^T = r(A^T)$	) if r is a number.		
3	$(A+B)^T = A$	$A^T + B^T$ .		
Exerc	cise 7 (solution	ו)		
	$(A^ op+2E$	$(B^{ op})^{ op}=2A,  ext{ the original equat}$	cion	
	$A^{ op}$ +	$2B^{ op} = (2A)^{ op}$ by part (1)		
	$A^{ op}$ +	$2B^ op=2(A^ op)$ by part (2) or	(3)	
		$2B^{\top} = A^{\top}$ , after adding $-A^{\top}$	$^ op$ to both sides	

$$B^{\top} = \frac{1}{2}(A^{\top}) = \left(\frac{1}{2}A\right)^{T} \text{ by part (2)}$$
$$B = \left[\frac{1}{2}A\right] \text{ by part (1)}$$

To verify your solution, plug in  $B = \frac{1}{2}A$  back into the original equation.

Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00	00	0000

Let's return to the properties that aren't true.

Recall: Matrix multiplication is not commutative

AB may not equal BA.

So, although we can pick the order in which we multiply pairs of neighbors (due to associativity), for example,

ABCD = A((BC)D)

we cannot rearrange the matrices.

ABCD may not equal ADBC

Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00	00	0000

## Example

Convince yourself that

$$egin{array}{ccc} 1 & -1 \ -1 & 1 \end{array} egin{bmatrix} 1 & -1 \ 1 & -1 \end{bmatrix} = egin{bmatrix} 0 & 0 \ 0 & 0 \end{bmatrix}$$

Warning: The product of two non-zero matrices might be zero

AB = 0 doesn't always imply A = 0 or B = 0.

This never happens for numbers: if ab = 0, then a = 0 or b = 0.

Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00	00	0000

There's a more general version of this phenomenon.

## Warning: Multiplication can't always be cancelled

Having AB = AC and  $A \neq 0$  doesn't guarantee B = C.

This never happens for numbers: if ab = ac and  $a \neq 0$ , then b = c.

## Exercise 8

Compute the following.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00	00	0000

There's a more general version of this phenomenon.

## Warning: Multiplication can't always be cancelled

Having AB = AC and  $A \neq 0$  doesn't guarantee B = C.

This never happens for numbers: if ab = ac and  $a \neq 0$ , then b = c.

## Exercise 8

Compute the following.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
 and  $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ 

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 + 7 & 1 \\ -1 & 2 + 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Distributivity	Associativity	Transpose reverses multiplication order	Solving a matrix equation	Warning
0	000	00	00	0000

## Recap

- Some familiar properties of arithmetic hold for matrices.
- Some familiar properties don't!

...wait, what happened to division?

## Next time

Why division doesn't always work, and how to perform division when it does work.