Lecture 4b

Matrix Multiplication (properties of matrix arithmetic)

Properties of matrix arithmetic

Last Time

Matrix multiplication violates one of the basic rules of traditional arithmetic: matrix multiplication does not always commute.

Goal

What familiar properties does matrix arithmetic have?

Matrix addition is associative and commutative

Associativity of addition

$$A + (B + C) = (A + B) + C$$

So, we can write $A + B + C$ to mean

"$A + (B + C)$ or $(A + B) + C$, your choice"

Commutativity of addition

$$A + B = B + A$$

As a result, we can rearrange sums of matrices however we want.

$$\begin{aligned} A + B + C &= C + A + B \\ &= B + C + A \\ &= C + B + A \end{aligned}$$

Multiplication distributes over addition

$$A(B + C) = AB + AC \qquad (B + C)A = BA + CA$$

Exercise 4

Compute $Av + Aw$, where $A := \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $v := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $w := \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Note: It takes less effort to compute $A(v + w)$ than to compute $Av + Aw$.

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Note: It takes less effort to compute $A(v+w)$ than to compute $Av + Aw$.

$$v + w = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$3 \times 2 \quad 2 \times 1 \qquad 3 \times 1$

$$A(v+w) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 + 1\cdot 3 \\ -1\cdot 3 + 1\cdot 3 \\ 3\cdot 3 + 0\cdot 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix}$$

Matrix multiplication is associative

Associativity of multiplication

$$A(BC) = (AB)C$$

So, we can write ABC to mean "A(BC) or (AB)C, your choice".

Exercise 5

Check that $(AB)C = A(BC)$, where

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \qquad B := \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix} \qquad C := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Compute $A(BC)$ and $(AB)C$. Which took more computation?

Compute $\boxed{A(BC)}$ and (AB)C. Which took more computation?

$$\underline{BC} = \overset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \overset{3\times 1}{\begin{bmatrix} 1.1+0.2+0.3 \\ -1.1+2.2+0.3 \\ 2.1+-1.2+2.3 \end{bmatrix}} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$A(BC) = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}}_{BC} = \begin{bmatrix} 1.1+2.3+3.6 \\ 0.1+1.3+2.6 \\ 0.1+0.3+-1.6 \end{bmatrix} = \begin{bmatrix} 25 \\ 15 \\ -6 \end{bmatrix}$$

$$\underline{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \underline{\begin{bmatrix} 5 & 1 & 6 \\ 3 & 0 & 4 \\ -2 & 1 & -2 \end{bmatrix}}$$

Took more time

$$(AB)C = \begin{bmatrix} 5 & 1 & 6 \\ 3 & 0 & 4 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5.1+1.2+6.3 \\ 3.1+0.2+4.3 \\ -2.1+1.2+-2.3 \end{bmatrix} = \begin{bmatrix} 25 \\ 15 \\ -6 \end{bmatrix}$$

So far

- Matrix addition is both associative and commutative
- Matrix multiplication is associative but is <u>not</u> commutative

Can you think of a familiar arithmetic operation (from elementary school) which is <u>not</u> associative?

Transpose reverses the order of multiplication

$$(AB)^\top = B^\top A^\top$$

Exercise 6

Check that $(AB)^\top = B^\top A^\top$ for $A := \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $B := \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

Exercise 6

Check that $(AB)^\top = B^\top A^\top$ for $A := \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $B := \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 1\cdot 2 & 2\cdot -1 + 1\cdot 2 \\ -1\cdot 1 + 1\cdot 2 & -1\cdot -1 + 1\cdot 2 \\ 3\cdot 1 + 0\cdot 2 & 3\cdot -1 + 0\cdot 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 3 & -3 \end{bmatrix}$$

$$(AB)^\top = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

3rd row is now 3rd col

$$B^\top = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \quad A^\top = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B^\top A^\top = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} \overset{(1,1)}{1\cdot 2 + 2\cdot 1} & \overset{(1,2)}{1\cdot -1 + 2\cdot 1} & \overset{(1,3)}{1\cdot 3 + 2\cdot 0} \\ \overset{(2,1)}{-1\cdot 2 + 2\cdot 1} & \overset{(2,2)}{-1\cdot -1 + 2\cdot 1} & \overset{(2,3)}{-1\cdot 3 + 2\cdot 0} \end{bmatrix}}_{2\times 3} = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

Theorem

Suppose A and B are matrices.

1. $(A^T)^T = A$.
2. $(rA)^T = r(A^T)$ if r is a number.
3. $(A + B)^T = A^T + B^T$.

Algebra with matrices

Using these rules, we can rearrange expressions involving matrices.

Exercise 7

Solve the following expression for B. Write B in terms of A.

$$(A^\top + 2B^\top)^\top = 2A$$

Theorem

1. $(A^T)^T = A.$
2. $(rA)^T = r(A^T)$ *if r is a number.*
3. $(A+B)^T = A^T + B^T.$

Exercise 7 (solution)

$$
\begin{aligned}
(A^\top + 2B^\top)^\top &= 2A, \text{ the original equation} \\
A^\top + 2B^\top &= (2A)^\top \text{ by part (1)} \\
A^\top + 2B^\top &= 2(A^\top) \text{ by part (2) or (3)} \\
2B^\top &= A^\top, \text{ after adding } -A^\top \text{ to both sides} \\
B^\top &= \frac{1}{2}(A^\top) = \left(\frac{1}{2}A\right)^T \quad \text{by part (2)} \\
B &= \boxed{\frac{1}{2}A} \text{ by part (1)}
\end{aligned}
$$

To verify your solution, plug in $B = \frac{1}{2}A$ back into the original equation.

Let's return to the properties that aren't true.

Recall: Matrix multiplication is not commutative

AB may not equal BA.

So, although we can pick the order in which we multiply pairs of neighbors (due to associativity), for example,

$$ABCD = A((BC)D)$$

we cannot rearrange the matrices.

$$ABCD \text{ may not equal } ADBC$$

Example

Convince yourself that

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Warning: The product of two non-zero matrices might be zero

$\mathrm{AB} = 0$ doesn't always imply $\mathrm{A} = 0$ or $\mathrm{B} = 0$.

This never happens for numbers: if $ab = 0$, then $a = 0$ or $b = 0$.

There's a more general version of this phenomenon.

Warning: Multiplication can't always be cancelled

Having $AB = AC$ and $A \neq 0$ doesn't guarantee $B = C$.

This never happens for numbers: if $ab = ac$ and $a \neq 0$, then $b = c$.

Exercise 8

Compute the following.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

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This never happens for numbers: if $ab = ac$ and $a \neq 0$, then $b = c$.

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$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.1 + {-1}.2 \\ -1.1 + 1.2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.{-2} + {-1}.{-1} \\ -1.{-2} + 1.{-1} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Recap

- Some familiar properties of arithmetic hold for matrices.
- Some familiar properties don't!

...wait, what happened to division?

Next time

Why division doesn't always work, and how to perform division when it does work.