## Lecture 4 a

## Matrix Multiplication

## Recall from lecture 3b: Multiplying a matrix and a vector

Given a $m \times n$-matrix A and an $n$-vector $\mathrm{v}, \mathrm{Av}$ is the $m$-vector whose $i$ th entry is the dot product of the $i$ th row of $A$ with $v$.

Example: Multiplying a $3 \times 3$ matrix and a 3 -vector

$$
\left[\begin{array}{ccc}
2 & -3 & 4 \\
-2 & 2 & -3 \\
-3 & 4 & 2
\end{array}\right]\left[\begin{array}{c}
2 \\
1 \\
-1
\end{array}\right]=\left[\begin{array}{c}
2 \cdot 2+(-3) \cdot 1+4 \cdot(-1) \\
-2 \cdot 2+2 \cdot 1-3 \cdot(-1) \\
-3 \cdot 2+4 \cdot 1+2 \cdot(-1)
\end{array}\right]=\left[\begin{array}{c}
3 \\
1 \\
-4
\end{array}\right]
$$



Goal for lecture 4a
Multiplying a matrix by a marix.

## Denoting a specific entry of a matrix

The $(i, j)$ th entry of a matrix A is the entry in the $i$ th row and the $j$ th column, and is denoted $A_{i, j}$ or $a_{i, j}$.

## Example

The $(2,3)$ th entry of the following matrix is 2 .

$$
\left[\begin{array}{cccc}
1 & -1 & 2 & -2 \\
0 & 1 & 2 & 0
\end{array}\right]
$$

This notation counts down then over, unlike Cartesian coordinates.

$$
\left[\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & (1,4 \\
(2,1) & (2,2) & (2,3) & (2,4)
\end{array}\right]
$$

## Matrix multiplication: the idea

Given matrices $A$ and $B$, the product $A B$ is the matrix whose $(i, j)$ th entry is the dot product of row $i$ of $A$ and column $j$ of $B$.

To compute the $(i, j)$-entry of $A B$, do:
Go across row $i$ of $A$, and down column $j$ of $B$, multiply corresponding entries, and add the results.


The rows of $A$ must be the same length as the columns of $B$.

## Matrix multiplication: the formula

The product AB is a matrix whose $(i, j)$ th entry is the sum of the product of the row $i$ of $A$ with the column $j$ of $B$.

$$
\begin{aligned}
& {\left[\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
a_{i, 1} & a_{i, 2} & \cdots & a_{i, n} \\
\vdots & \vdots & \vdots & \vdots
\end{array}\right]\left[\begin{array}{ccc}
\cdots & b_{1, j} & \cdots \\
\cdots & b_{2, j} & \cdots \\
\cdots & \vdots & \cdots \\
\cdots & b_{n, j} & \cdots
\end{array}\right]} \\
& =\left[\begin{array}{ccc}
\ddots & \vdots & \\
\cdots & a_{i, 1} b_{1, j}+a_{i, 2} b_{2, j}+\ldots+a_{i, n} b_{n, j} & \cdots \\
\ddots & \vdots & \ddots
\end{array}\right]
\end{aligned}
$$

## Exercise 1

Compute the $(1,3)$ - and (2,4)-entries of $A B$ where

$$
A=\left[\begin{array}{rrr}
3 & -1 & 2 \\
0 & 1 & 4
\end{array}\right] \text { and } B=\left[\begin{array}{rrrr}
2 & 1 & 6 & 0 \\
0 & 2 & 3 & 4 \\
-1 & 0 & 5 & 8
\end{array}\right]
$$

Then compute $A B$.

The $(1,3)$ th-entry of $A B$ is the dot product of row 1 of $A$ and column 3 of $B$ (highlighted in the following display), computed by multiplying corresponding entries and adding the results.

$$
\begin{gathered}
{\left[\begin{array}{rrr}
3 & -1 & 2 \\
0 & 1 & 4
\end{array}\right]\left[\begin{array}{rrrr}
2 & 1 & 6 & 0 \\
0 & 2 & 3 & 4 \\
-1 & 0 & 5 & 8
\end{array}\right]} \\
(1,3) \text {-entry }=3 \cdot 6+(-1) \cdot 3+2 \cdot 5=25 \\
{\left[\begin{array}{rrrr}
\cdot & \cdot & \mathbf{2 5} & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{array}\right.}
\end{gathered}
$$

Similarly, the (2,4)-entry of $A B$ involves row 2 of $A$ and column 4 of $B$.

$$
\begin{gathered}
{\left[\begin{array}{rrr}
3 & -1 & 2 \\
0 & 1 & 4
\end{array}\right]\left[\begin{array}{rrrr}
2 & 1 & 6 & 0 \\
0 & 2 & 3 & 4 \\
-1 & 0 & 5 & 8
\end{array}\right]} \\
(2,4) \text {-entry }=0 \cdot 0+1 \cdot 4+4 \cdot 8=36 \\
{\left[\begin{array}{rrrr}
\cdot & \cdot & 25 & \cdot \\
\cdot & \cdot & 36
\end{array}\right]}
\end{gathered}
$$

Pause the video and compute the rest of the entries of $A B$.

$$
A B=\left[\begin{array}{rrr}
A & B & A B \\
0 & 1 & 4
\end{array}\right]\left[\begin{array}{rrrr}
2 & 1 & 6 & 0 \\
0 & 2 & 3 & 4 \\
-1 & 0 & 5 & 8
\end{array}\right]=\left[\begin{array}{rrrr}
4 & 1 & 25 & 12 \\
-4 & 2 & 23 & 36
\end{array}\right]
$$

Dimensions $2 \times 3 \quad 3 \times 4 \quad 2 \times 4$

## Dimensions must match!

For $A B$ to exist, we need

$$
\operatorname{width}(A)=\operatorname{height}(B)
$$

## Size of a product

If $A$ is $m \times n$ and $B$ is $n \times p$, then $A B$ is $m \times p$.
It can be helpful to think of the middle dimensions as cancelling:

$$
(m \times n)(n \times p)=m \times p
$$

If the middle dimensions don't agree, the product doesn't exist!

## Exercise 2(i)

Compute the product

$$
\left[\begin{array}{ccc}
1 & 2 & -1 \\
1 & 0 & 2
\end{array}\right]\left[\begin{array}{cc}
2 & 1 \\
-1 & 3 \\
0 & 1
\end{array}\right]
$$

Exercise 2(i) (solution)

$$
\begin{aligned}
\text { Compute the product } \\
\left.\begin{array}{ccc}
{\left[\begin{array}{lll}
1 & 2 & -1 \\
1 & 0 & 2
\end{array}\right]} \\
A & {\left[\begin{array}{c}
2 \\
-1 \\
0 \\
3 \\
1
\end{array}\right]} & =\left[\begin{array}{ll}
1.2+2 .-1+-1.0 & 1.1+2.3+(-1) .1 \\
1.2+0 .-1+2.0 & 1.1+0.3+2.1
\end{array}\right] \\
& =\left[\begin{array}{ll}
0 & 6 \\
2 & 3
\end{array}\right]
\end{array}\right]
\end{aligned}
$$

Dimensions

$$
A
$$

$$
B
$$

$A B$

$$
2 \times 3 \quad 3 \times 2 \quad 2 \times 2
$$

## Exercise 2(ii)

Compute the product

$$
\left[\begin{array}{cc}
2 & 1 \\
-1 & 3 \\
0 & 1
\end{array}\right]\left[\begin{array}{ccc}
1 & 2 & -1 \\
1 & 0 & 2
\end{array}\right]
$$

## Exercise 2(ii) (solution)

Compute the product

$$
\begin{aligned}
& {\left[\begin{array}{cc}
2 & 1 \\
-1 & 3 \\
0 & 1
\end{array}\right]\left[\begin{array}{ccc}
1 & 2 & -1 \\
1 & 0 & 2
\end{array}\right]=} \\
& A \\
& 3 \times 2
\end{aligned}
$$

$$
=\left[\begin{array}{ccc}
3 & 4 & 0 \\
2 & -2 & 7 \\
1 & 0 & 2
\end{array}\right] \begin{aligned}
& A B \\
& 3 \times 3
\end{aligned}
$$

## Order matters in matrix multiplication!

We have seen that

$$
\left[\begin{array}{ccc}
1 & 2 & -1 \\
1 & 0 & 2
\end{array}\right]\left[\begin{array}{cc}
2 & 1 \\
-1 & 3 \\
0 & 1
\end{array}\right]=\left[\begin{array}{ll}
0 & 6 \\
2 & 3
\end{array}\right] \neq\left[\begin{array}{cc}
2 & 1 \\
-1 & 3 \\
0 & 1
\end{array}\right]\left[\begin{array}{ccc}
1 & 2 & -1 \\
1 & 0 & 2
\end{array}\right]
$$

This means that matrix multiplication is not always commutative!
That is, $A B$ is usually not equal to $B A$, though it can happen.

## Exercise 3

Check whether the following matrices commute.

$$
\left[\begin{array}{cc}
1 & -2 \\
1 & 2
\end{array}\right] \text { and }\left[\begin{array}{cc}
1 & 6 \\
-3 & -2
\end{array}\right]
$$

## Order matters in matrix multiplication!

Matrix multiplication is not always commutative!
That is, $A B$ is usually not equal to $B A$, though it can happen.

## Exercise 3 (solution)

Check whether the following matrices commute.

Answer:

$$
\left[\begin{array}{cc}
1 & -2 \\
1 & 2
\end{array}\right] \text { and }\left[\begin{array}{cc}
1 & 6 \\
-3 & -2
\end{array}\right] \quad \begin{gathered}
\text { Yes, they } \\
\text { commute }
\end{gathered}
$$

$\left[\begin{array}{cc}1 & -2 \\ 1 & 2\end{array}\right]\left[\begin{array}{cc}1 & 6 \\ -3 & -2\end{array}\right]=\left[\begin{array}{ll}1.1+-2 .(-3) & 1.6+-2 .-2 \\ 1.1+2 .-3 & 1.6+2 .-2\end{array}\right]=\left[\begin{array}{ll}7 & 10 \\ \hline-5 & 2\end{array}\right]$
$\left[\begin{array}{cc}1 & 6 \\ -3 & -2\end{array}\right]\left[\begin{array}{cc}1 & -2 \\ 1 & 2\end{array}\right]=\left[\begin{array}{cc}1.1+6.1 & 1 .-2+6.2 \\ -3.1+-2.1 & -3 .-2+-2.2\end{array}\right]=\left[\begin{array}{ll}7 & 10 \\ -5 & 2\end{array}\right.$

