Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Lecture 4a

Matrix Multiplication

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Recall from lecture 3b: Multiplying a matrix and a vector

Given a $m \times n$ -matrix A and an *n*-vector v, Av is the *m*-vector whose *i*th entry is the dot product of the *i*th row of A with v.

Example: Multiplying a 3×3 matrix and a 3-vector

$$\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + (-3) \cdot 1 + 4 \cdot (-1) \\ -2 \cdot 2 + 2 \cdot 1 - 3 \cdot (-1) \\ -3 \cdot 2 + 4 \cdot 1 + 2 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

Goal for lecture 4a

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Multiplying a matrix by a marix.

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Denoting a specific entry of a matrix

The (i,j)th entry of a matrix A is the entry in the *i*th row and the *j*th column, and is denoted A_{*i*,*j*} or $a_{i,j}$.

Example

The (2,3)th entry of the following matrix is **2**.

$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

This notation counts down then over, unlike Cartesian coordinates.

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \end{bmatrix}$$

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Matrix multiplication: the idea

Given matrices A and B, the product AB is the matrix whose (i, j)th entry is the dot product of row *i* of A and column *j* of B.

To compute the (i, j)-entry of AB, do:

Go *across* row *i* of *A*, and *down* column *j* of *B*, multiply corresponding entries, and add the results.



The rows of A must be the same length as the columns of B.

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Matrix multiplication: the formula

The product AB is a matrix whose (i, j)th entry is the sum of the product of the row *i* of A with the column *j* of B.

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_{1,j} & \cdots \\ \cdots & b_{2,j} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & b_{n,j} & \cdots \end{bmatrix}$$
$$= \begin{bmatrix} \ddots & & & \ddots \\ \cdots & a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \dots + a_{i,n}b_{n,j} & \cdots \\ \ddots & & \vdots & & \ddots \end{bmatrix}$$

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Exercise 1

Compute the (1, 3)- and (2, 4)-entries of AB where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}$$

Then compute *AB*.

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Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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The (1,3)th-entry of AB is the dot product of row 1 of A and column 3 of B (highlighted in the following display), computed by multiplying corresponding entries and adding the results.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}$$

$$(1,3)$$
-entry = $3 \cdot 6 + (-1) \cdot 3 + 2 \cdot 5 = 25$

$$\left[\begin{array}{cccc} \cdot & \cdot & \mathbf{25} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right]$$

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
0	000000	0	000000

Similarly, the (2, 4)-entry of *AB* involves row 2 of *A* and column 4 of *B*.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}$$

$$(2, 4)$$
-entry = $0 \cdot 0 + 1 \cdot 4 + 4 \cdot 8 = 36$

Pause the video and compute the rest of the entries of AB.

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{bmatrix}$$

Dimensions 2 × 3 3×4 2×4

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Dimensions must match!

For AB to exist, we need

$$\operatorname{width}(\mathsf{A}) = \operatorname{height}(\mathsf{B})$$

Size of a product

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

It can be helpful to think of the middle dimensions as cancelling:

$$(m \times p)(n \times p) = m \times p$$

If the middle dimensions don't agree, the product doesn't exist!

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Exercise 2(i)

Compute the product

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}$$

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Exercise 2(i) (solution) Compute the product $\begin{bmatrix} 2\\ -1\\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2\\ -1\\ 0 \end{bmatrix}$ $\begin{bmatrix} 1.2 + 2. -[+ -1.0 & 1.1 + 2.3 + (-0.] \\ 1.2 + 0. -[+ 2.0 & 1.1 + 0.3 + 2.] \end{bmatrix}$ $\begin{pmatrix} 1\\ 3\\ 1 \end{pmatrix}$ A $= \begin{bmatrix} 0 & 6 \\ 2 & 3 \end{bmatrix}$ AB 2×2 3 15 3 x 2 Dimensions 2×3

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Exercise 2(ii)

Compute the product

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Exercise 2(ii) (solution)

Compute the product $= \begin{bmatrix} 3 & 4 & 0 \\ 2 & -2 & 7 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{A B} 3 \times 3$

Denoting an entry	Matrix multiplication	Dimensions must match!	Matrix multiplication is not commutative
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Order matters in matrix multiplication!

We have seen that

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

This means that matrix multiplication is not always commutative!

That is, AB is usually not equal to BA, though it can happen.

Exercise 3

Check whether the following matrices commute.

$$\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 6 \\ -3 & -2 \end{bmatrix}$$

Order matters in matrix multiplication!

Matrix multiplication is not always commutative!

That is, AB is usually not equal to BA, though it can happen.

Exercise 3 (solution) Check whether the following matrices commute.

 $\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 6 \\ -3 & -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot (-3) & 1 \cdot 6 + 2 \cdot -2 \\ 1 \cdot 1 + 2 \cdot (-3) & 1 \cdot 6 + 2 \cdot -2 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ -5 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 6 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 6 \cdot 1 & 1 \cdot -2 + 6 \cdot 2 \\ -3 \cdot 1 + 2 \cdot (-3 \cdot -2 + 2 \cdot 2) \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ -5 & 2 \end{bmatrix}$$

Answer: Yes, they commute