Lecture 3a

Matrices and Vectors

Plan

Introduce vectors, matrices, and matrix arithmetic.

Definition: Vectors

A **vector** is a column of numbers, surrounded by brackets.

Examples of vectors

$$\begin{bmatrix}2\\3\\4\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}-3\end{bmatrix} \quad \begin{bmatrix}-3\\0\\1\\2\end{bmatrix}$$

is a 4-vector.
Its size is 4,
and its entries
are -3, 0, 1, and 2

- The entries are sometimes called **coordinates**.
- The **size** of a vector is the number of entries.
- An ***n*-vector** is a vector of size n.
- We sometimes call these **column vectors**, to distinguish them from **row vectors** (when we write the numbers horizontally).

An example of a row vector: [2, 3, 4]

The purpose of vectors

The purpose of vectors is to collect related numbers together and work with them as a single object.

Example of a vector: Latitude and longitude

A point on the globe can be described by two numbers: the latitude and longitude. These can be combined into a single vector:

$$\text{Position of Norman's train station} = \begin{bmatrix} 35.13124^\circ, -97.26343^\circ \end{bmatrix}$$

(row vector)

Example of a vector: solution of a linear system

A **solution** of a linear system is given in terms of values of variables, even though we think of this as one object:

$$x = 3, \quad y = 0, \quad z = -1$$

$$(x, y, z) = (3, 0, -1)$$

We can restate this as a (column) vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

Definition: Matrices

A **matrix** is a rectangular grid of numbers, surrounded by brackets.

Examples of matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

is a $\underbrace{2}_{\#\text{rows}} \times \underbrace{4}_{\#\text{cols}}$ matrix

- The individual numbers are called the **entries** of the matrix.
- The **height** of a matrix is the number of **rows**.
- The **width** of a matrix is the number of **columns**.
- An $m \times n$-**matrix** is a matrix of height m and width n.

The purpose of matrices

Matrices serve two distinct purposes.

1. To collect tables of related numbers together.
2. To describe linear transformations between vector spaces.

The second item should be totally mysterious right now!

Example: Several global positions

We can collect the latitude and longitude of several locations (e.g. Dale Hall, PHSC, and the Union) into a single matrix.

$$\begin{bmatrix} 35.204183^\circ & 35.209397^\circ & 35.209474^\circ \\ -97.446405^\circ & -97.447460^\circ & -97.444157^\circ \end{bmatrix}$$

Example: Economic mobility

We can make a matrix whose entries record the probability that the child born in the upper/middle/lower income bracket will grow up into the upper/middle/lower income bracket. [a]

$$\begin{bmatrix} 50\% & 20\% & 20\% \\ 40\% & 60\% & 30\% \\ 10\% & 20\% & 50\% \end{bmatrix}$$

For example, being born in the upper bracket gives a 10% chance of ending up in the lower bracket.

[a] These nice whole numbers are from Exercise 2.9.4 in the textbook, not based on actual data

Example: Vectors are matrices

Every n-vector is an $n \times 1$-matrix. For example,

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{is a 3-vector and a } 3 \times 1\text{-matrix}$$

Arithmetic with vectors and matrices

We define several arithmetic operations on matrices.

The sum of two matrices

If A and B are matrices of the same size, then $A + B$ is the matrix whose entries add the corresponding entries of A and B.

Variables denoting matrices are usually capitalized.

Example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 2 & 1 \\ 5 & 5 & 5 & 8 \end{bmatrix}$$

The sum of two matrices of different sizes does not make sense!

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$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

is not defined. (2×4 matrix) (2×3 matrix)

The sum of two matrices of different sizes does not make sense!

Scalar multiplication

If A is a matrix and r is a number, then rA is the matrix whose entries multiply the corresponding entry of A by r.

By 'number', we mean a real number unless otherwise specified.

Example

$$2 \cdot \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -2 & -6 \\ 0 & -2 & -4 & 0 \end{bmatrix}$$

Scalar is just a fancy word for 'number'. We are distinguishing scalar multiplication from matrix multiplication, which we'll learn about in another lecture.

Scalar multiplication distributes over addition:

$$r(A + B) = rA + rB$$

Example:

$$2\left(\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}+\begin{bmatrix}1 & 3 & -1 & -3\\0 & -1 & -2 & 0\end{bmatrix}\right)=2\cdot\begin{bmatrix}2 & 5 & 2 & 1\\5 & 5 & 5 & 8\end{bmatrix}$$

$$\parallel$$

$$2\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}+2\begin{bmatrix}1 & 3 & -1 & -3\\0 & -1 & -2 & 0\end{bmatrix}$$

We combine addition and scalar multiplication into one concept.

Linear combinations

A **linear combination** of matrices is an expression of the form.

$$r_1 A_1 + r_2 A_2 + \ldots + r_n A_n$$

where the $r_1, r_2, \ldots, r_n$ are numbers and $A_1, A_2, \ldots, A_n$ are matrices of the same size.

Example: a linear combination of a vector

The expression

$$\boxed{2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 6 \\ -5 \end{bmatrix}$$

← A linear combination of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

is called a linear combination.

We would say "The vector $\begin{bmatrix} 0 \\ 6 \\ -5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$."

Exercise 1: linear combination

Write the linear system $\begin{cases} 3x_1 + 2x_2 - 4x_3 = 0 \\ x_1 - 3x_2 + x_3 = 3 \\ x_2 - 5x_3 = -1 \end{cases}$ as a linear combination of a vector.

Answer:

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

Transpose

Given an $m \times n$-matrix A, the **transpose** $A^\top$ is the $n \times m$-matrix which reflects along the main diagonal.

Equivalently, the transpose exchanges rows and columns.

Example

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -1 & -2 & 0 \end{bmatrix}^\top = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ -1 & -2 \\ -3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^\top = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Properties of the transpose

$$(A + B)^\top = A^\top + B^\top, \quad (rA)^\top = r(A^\top), \quad (A^\top)^\top = A$$

Exercise 2

Write down the transpose of each of the following matrices.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

Answer:

$$A^T = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, \; B^T = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}, \; C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \text{ and } D^T = D.$$