Lecture 2b

Gaussian Elimination (con't)

Last time

- A definition for systems that can be 'easily solved': Row echelon form (REF)
- An algorithm for picking which elementary operation to use: Gaussian elimination
- Knowing when there are no solutions, a unique solution, or multiple solutions.

Note: When we say a system has a unique solution, this means the system has exactly one solution.

Today

- Rank of a matrix
- Homogeneous system

The number and location of leading 1s is a useful fact.

Rank of a matrix

The **rank** of a matrix is the number of leading 1s in any equivalent REF matrix.

To find the rank, we can use a sequence of elementary row operations (like Gaussian Elimination) to put it in REF.

Exercise 4

Let

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

Find the rank of M.

A matrix might be equivalent to more than one REF matrix, but they all have the same number of leading 1s!

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow[R_3 - 2R_2]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ An REF matrix equivalent to M

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[R_2 + R_3]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[-R_2]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ A different REF matrix also equivalent to M

Note: Both REF matrices have the same number of leading 1s

All matrices on this page (incl. M) are of rank $\boxed{2}$ because they are all equivalent to an REF matrix with two leading 1s.

The # of leading 1's cannot be bigger than # of columns

Bounds on rank

The rank of a matrix is at most the number of rows, and at most the number of columns.

Knowing the rank tells us 'how big' the solution set is.

Rank and solutions

Consider you have a linear system in n-many variables whose aug. matrix has rank r. If it's consistent, the set of solutions has $(n-r)$-many parameters. (aka. has at least a solution)

In particular, if it is consistent...

- it has a unique solution if $n = r$, and
- it has infinitely many solutions if $n > r$.

E.g. In Ex 4, $n=2$, $r=2$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{array}\right] \xrightarrow{\text{Gauss El.}} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$n-r=0$ parameters

i.e. this system has a unique solution.

# of variables = # of columns (not counting the right-most column)

systems of linear equations

Let's finish by considering an important class of SLEs.

Homogeneous systems of linear equations

A system is **homogeneous** if each constant term is 0.

Example

$$\begin{aligned} x + y + z &= 0 \\ x + 2y + 3z &= 0 \\ x + 3y + 5z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right]$$

For aug. matrices, homogeneous means the right column is all 0s.

A simple observation

A homogeneous SLE has a solution in which every variable is 0.

Therefore, every homogeneous SLE is consistent.

An easy fact

Elementary operations don't change homogeneity.

That is, if the right column is 0s, it will stay 0s.

A time-saver

When solving homogeneous SLEs, you can suppress the last column of the aug. matrix...just don't forget about the hidden column!

Exercise 5

a) Find all the solutions to the following system of linear equations.

$$x + y + z = 0$$
$$x + 2y + 3z = 0$$
$$x + 3y + 5z = 0$$

b) Find the rank of the corresponding aug. matrix

Reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[R_3-2R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note from instructor:
Column 3 does not have a leading 1,
so pick a parameter for the variable z

Let $z = t$

Eq. 2 : $y + 2z = 0$ implies $y + 2t = 0$, so $y = -2t$

Eq. 3 : $x + y + z = 0$ implies $x + (-2t) + t = 0$, so $x = t$

All (infinitely many) solutions are of the form
$(x, y, z) = (t, -2t, t)$, where t is a number

b) The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{bmatrix}$ is

the number of leading 1s in any REF matrix

equivalent to $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{bmatrix}$ (from the def in Slide #3).

Since $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is an REF matrix equivalent to $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{bmatrix}$,

and since there are two leading 1s in $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

the rank of $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{bmatrix}$ is 2.

Find all solutions to the following homogeneous system.

Exercise 6

$$
\begin{aligned}
x_1 - x_2 + 2x_3 - x_4 &= 0 \\
2x_1 + 2x_2 + x_4 &= 0 \\
3x_1 + x_2 + 2x_3 - x_4 &= 0
\end{aligned}
$$

Here, we have $n = 4$ variables.

Reduce the augmented matrix to reduced row-echelon form

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 0 & 4 & -4 & 3 & 0 \\ 0 & 4 & -4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Column $1, 2, 4$ have leading 1s (but not column 3), so x_3 is assigned as a parameter—say $x_3 = t$. Then the general solution (start from the bottom row) is

$$\begin{aligned} & & x_4 &= 0, \\ & & x_3 &= t \text{ (because we said so earlier)} \\ &\text{2nd Row :} & x_2 - x_3 = 0, \text{ so } x_2 &= x_3 = t, \\ &\text{1st Row :} & x_1 + x_3 = 0, \text{ so } x_1 &= -x_3 = -t. \end{aligned}$$

All (infinitely many) solutions are of the form $\boxed{(x_1, x_2, x_3, x_4) = (-t, t, t, 0)}$.
(Note: here, the aug. matrix has rank $r = 3$, and we have $n - r = 4 - 3 = 1$ parameter, which matches what we said in Slide #4.)

For example, some solutions are $(0, 0, 0, 0)$, for $t = 0$, and $(-1, 1, 1, 0)$, for $t = 1$.

Suggested Exercises from the textbook (same as for lecture 2a):

Section 1.1, Exercises 7, 9, 11, and 14
Section 1.2, Exercises 3, 7, and 16