Lecture 2a

Gaussian Elimination

Last time

- A general problem (finding solutions to a system of linear equations)
- Tools for simplifying that problem (elementary operations)
- A vague strategy (eliminate variables)
- A time-saving notation (augmented matrices)

Today

- A definition for systems that can be 'easily solved'.
- An algorithm for picking which elementary operation to use.
- Dealing with no solutions or multiple solutions.

Row echelon form (REF)

A matrix is in **row echelon form (REF)** if...

- the first non-zero entry in each row is a 1 (a **leading** 1), and
- each row begins with more 0s than the row above it
(or all 0s if this is impossible).

Examples

The following are in row echelon form, with leading 1s in **bold**.

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & 2 & 3 \\ 0 & 0 & \mathbf{1} & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} \mathbf{1} & 2 & 3 & 1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & \mathbf{1} \end{array}\right] \quad \left[\begin{array}{ccc|c} \mathbf{1} & 2 & 3 & 0 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row echelon form (REF)

A matrix is in **row echelon form (REF)** if...

1 • the first non-zero entry in each row is a 1 (a **leading** 1), and

2 • each row begins with more 0s than the row above it (or all 0s if this is impossible).

Non-examples

The following are not in row echelon form, see the **bold numbers**.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & \mathbf{4} & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{array}\right]$$

Additional properties of REF matrices

- Any rows of zeroes must be at the bottom.
- Each leading 1 is to the right of the leading 1 above it.
- Each column can have at most one leading 1.

If an augmented matrix is in REF, the system is easy to solve!

Solving a system of linear equations in REF, Case 1

Suppose any of the following equivalent conditions hold:

- There is a leading 1 in the right column.
- There is a row $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$.
- The corresponding system has equation $0 = 1$.

Then the system of linear equations is **inconsistent**.

Otherwise, as we will see, the system is always consistent.

Example

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] \longmapsto \begin{aligned} x + 2y + 3z &= 1 \\ z &= 2 \\ 0 &= 1 \end{aligned}$$

Therefore, the system is inconsistent.

Solving a system of linear equations in REF, Case 2

If there is a leading 1 in each column except the right one...

- Turn the matrix into a system of linear equations.
- Starting at the bottom equation and going up, solve each equation for the variable with the leading 1.

This always produces the unique solution to the system.

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \mapsto \begin{aligned} x + y + z &= 1 \\ y + 2z &= 3 \\ z &= 2 \end{aligned} \mapsto \begin{aligned} x &= 0 \\ y &= -1 \\ z &= 2 \end{aligned}$$

So, $(x, y, z) = (0, -1, 2)$ is the unique solution to the system.

Solving a system of linear equations in REF, Case 3

If some columns don't have leading 1s ...

- Turn the matrix into a system of linear equations.
- Pick a parameter for each variable whose column doesn't have a leading 1. (The letters s and t are popular choices)
- Starting at the bottom equation and going up, solve each equation for the variable with the leading 1.

Every solution to the system will be of this form.

Exercise 1

Find all solutions to the system corresponding to the aug. matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Turn matrix into a system of linear equations

$$x + 2y + 3z = 0$$
$$z = 2$$
$$0 = 0$$

- Column for y has no leading 1
- Let $y = t$

Starting at the bottom equation and going up, solve each equation for the variable with the leading 1.

$$x + 2t + 3(2) = 0 \implies x = -2t - 6$$
$$z = 2$$

- The solutions of the system are all triples of the form $\boxed{(-2t-6, t, 2)}$

Solving systems of linear equations in REF (summary)

- If there is a leading 1 in the right column $\Rightarrow$ inconsistent.
- Else, turn the matrix into a system of linear equations.
- Pick a parameter for each variable whose column doesn't have a leading 1.
- Starting at the bottom equation and going up, solve each equation for the variable with the leading 1.

This process is called **back substitution**, since you are effectively plugging values from later equations 'back' into earlier ones.

Intuitively, each equation is used to find the value of the variable with the leading 1, and any leftover variables can be freely chosen.

Recall the following operations on an augmented matrix that preserve the solutions of the corresponding system.

The elementary row operations

- Exchange two rows.
- Multiply one row by a non-zero number.
- Add a multiple of one row to another row.

If we can use elementary row operations to put a matrix into REF, **then** we can easily find solutions to the original system.

Gaussian Elimination: The Idea

Starting with the top row and going down, create a leading 1 in the leftmost possible column, kill all entries below it, and repeat.

Exercise 2

Put the following augmented matrix into REF, and find all solutions to the corresponding system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & -1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & -1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

Make all other entries in Column 1 zero

$$\begin{array}{c} \\ -2R_1+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

Turn the leading nonzero entry in row 2 into the number 1 by multiplying row 2 by a number

$$\begin{array}{c} \\ -1R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

Make all entries below this zeros

$$\begin{array}{c} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

Turn the leading nonzero entry in row 2 into the number 1 by multiplying row 2 by a number

$$\begin{array}{c} \\ \\ \frac{1}{3}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

Gaussian Elimination: The Algorithm

1. If the matrix consists entirely of zeroes, stop.
2. Else, find the leftmost column with a non-zero entry. If the top row has a zero in that column, fix this by swapping rows.
3. Turn this into a leading 1 by multiplying the row by a number.
4. Make each entry below the leading 1 zero by subtracting multiples to the top row.
5. The top row is now fixed. Repeat these steps on the remaining rows, ignoring the top row.

- Easy to use on a computer

We can now find every solution to every system of linear equations!

How to find every solution to a system of linear equations

- Turn it into an augmented matrix.
- Put it into REF using **Gaussian Elimination**.
- Turn it back into a system of linear equations and find all solutions using **back substitution**.

- If you encounter a leading 1 in the right column, you can stop immediately because there are no solutions.

Exercise 3

Find all solutions to the following system.

$$\begin{aligned} 3x + y - 4z &= -1 \\ x + 10z &= 5 \\ 4x + y + 6z &= 1 \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 1 & -4 & -1 \\ 1 & 0 & 10 & 5 \\ 4 & 1 & 6 & 1 \end{array}\right]$$

Create the first leading 1 by interchanging rows 1 and 2

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & 1 & 6 & 1 \end{array}\right]$$

Replace $R2$ with $-3R1 + R2$. Replace $R3$ with $-4R1 + R3$. Get

$$\begin{array}{r} \\ -3R_1+R_2 \\ -4R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 1 & -34 & -19 \end{array}\right]$$

Now subtract R2 from R3 to obtain

$$\begin{array}{r} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

We've only applied elementary row operations, so the following reduced system

$$\begin{aligned} x \quad + 10z &= 5 \\ y - 34z &= -16 \\ 0 &= -3 \end{aligned}$$

is equivalent to the original system. But this last system has no solution (the last eq. requires that x, y and z satisfy $0x + 0y + 0z = -3$, and no such numbers exist).

The three possible cases for back substitution tell us a nifty fact.

Theorem: Number of Solutions to an SLE

A system of linear equations has **0**, **1**, or **∞-many** solutions.

The location of the leading 1s in the REF tells us which case it is.

Suggested exercise

Solve the following system of equations.

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 \phantom{{}+3x_4} &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

Answer explanation is given on Example 1.2.3 pg 13 in the textbook (copied to the next two slides).

Solution.

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right]$$

Subtracting twice row 1 from row 2 and subtracting row 1 from row 3 gives

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

Now subtract row 2 from row 3 and multiply row 2 by $\frac{1}{3}$ to get

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is in row-echelon form, and we take it to reduced form by adding row 2 to row 1:

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding reduced system of equations is

$$\begin{aligned} x_1 - 2x_2 \quad + \quad x_4 &= 2 \\ x_3 - 2x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

The leading ones are in columns 1 and 3 here, so the corresponding variables x_1 and x_3 are called leading variables. Because the matrix is in reduced row-echelon form, these equations can be used to solve for the leading variables in terms of the nonleading variables x_2 and x_4. More precisely, in the present example we set $x_2 = s$ and $x_4 = t$ where s and t are arbitrary, so these equations become

$$x_1 - 2s + t = 2 \quad \text{and} \quad x_3 - 2t = 1$$

Finally the solutions are given by

$$\begin{aligned} x_1 &= 2 + 2s - t \\ x_2 &= s \\ x_3 &= 1 + 2t \\ x_4 &= t \end{aligned}$$

where s and t are arbitrary.

Suggested Exercises from the textbook and Student Solution Manual:

Section 1.1, Exercises 7, 9, 11, and 14
Section 1.2, Exercises 3, 7, and 16