Lecture 1a

Systems of linear equations

The story of this class begins with the study of **linear equations**.

A linear equation

A **linear equation** in variables $x_1, x_2, \ldots, x_n$ is of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

for some numbers $a_1, a_2, \ldots, a_n, b$. The $a_1, a_2, \ldots, a_n$ are called the **coefficients**, and b is called the **constant** term.

Examples of linear equations

A linear equation in the variables x and y:

$$2x - 3y = 7$$

A linear equation in the variables x, y, and z:

$$x + .5y + \pi z = \sqrt{3}$$

A linear equation in the variables a, b, and c:

Not a linear equation

$$a^2 + 3b = 1$$

$$a + 3b - 6c = 1$$

Note: each variable in a linear equation occurs to the first power only.

A solution of a linear equations

A **solution** to **a** linear equation

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

is a choice of values for the variables for which the equation holds.

Examples of solutions

Consider the linear equation $2x - 3y = 7$.
The pair of values $x = 5$ and $y = 2$ is not a solution, because

$$2 \cdot 5 - 3 \cdot 2 \neq 7 \quad \mathcal{X}$$

The pair of values $x = 5$ and $y = 1$ is a solution, because

$$2 \cdot 5 - 3 \cdot 1 = 7 \quad \checkmark$$

The pair of values $(x, y) = (3.5, 0)$ is also a solution:

$$2 \cdot (3.5) - 3 \cdot 0 = 7 \quad \checkmark$$

A single linear equation typically has lots of solutions! The (x, y) notation lets us describe multiple values at once.

A solution of a linear equations

A **solution** to **a** linear equation

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

is a choice of values for the variables for which the equation holds.

More Examples of solutions

Consider the linear equation $2x - 3y + 5z = 7$.
The triple of values $x = 5$, $y = 1$, $z = 0$ is a solution, because

$$2 \cdot 5 - 3 \cdot 1 + 5 \cdot 0 = 7 \quad \checkmark$$

The triple of values $(x, y) = (3.5, 5, 3)$ is also a solution:

$$2 \cdot (3.5) - 3 \cdot 5 + 5 \cdot 3 = 7 \quad \checkmark$$

A single linear equation typically has lots of solutions! The (x, y, z) notation lets us describe multiple values at once.

Linear equations like to 'travel in packs'.

A system of linear equations

A **system of linear equations** (sometimes called a **linear system**) is a set of linear equations in the same variables.

Examples of systems of linear equations

A system of 2 linear equations in the two variables x and y:

$$
\begin{aligned}
2x - 3y &= 7 \\
x + y &= 6
\end{aligned}
$$

A system of 2 linear equations in the three variables a, b, and c:

$$
\begin{aligned}
a + 3b - 6c &= 1 \\
3a - 2b + 1c &= -3
\end{aligned}
$$

A solution of a system of linear equations

A sequence of numbers is called **a solution to a system** of linear equations if it is a solution to *every* equation in the system.

Examples of solutions

Consider the linear system

$$\left.\begin{aligned} 2x - 3y &= 7 \\ x + y &= 6 \end{aligned}\right\} \text{ is consistent}$$

The pair of values $x = 5$ and $y = 1$ is a solution, because

$$2 \cdot 5 - 3 \cdot 1 = 7 \quad \checkmark$$
$$5 + 1 = 6 \quad \checkmark$$

The pair of values $(x, y) = (3.5, 0)$ is not a solution:

$$2 \cdot (3.5) - 3 \cdot 0 = 7 \quad \checkmark$$
$$3.5 + 0 \neq 6 \quad \text{✗}$$

A system of linear equations need not have any solutions!

A linear system with no solutions

inconsistent
$$\begin{cases} x + 2y = 1 \\ 2x + 4y = 1 \end{cases}$$

Suppose there is (a,b) so that
$a + 2b = 1.$
$2a + 4b = 2.$

If we pick values of x and y so that $x + 2y = 1$, then multiplying both sides by 2 shows that

$$2x + 4y = 2$$

$2a + 4b = 1$

Not possible

This cannot be true at the same time that $2x + 4y = 1$!

A linear system with no solutions is called **inconsistent**.
A linear system with solutions is called **consistent**.

A system of linear equations can have many solutions!

Exercise 1: A linear system with infinitely-many solutions

Show that $(x, y) = (1 - 2s, s)$ is a solution of the following linear system, for arbitrary values of s.

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 2 \end{aligned}$$

A variable used to describe a bunch of solutions at once (like s above) is called a **parameter**.

Exercise 1

Show that $(x,y) = (1-2s, s)$, for any s, is a solution of the system

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 2 \end{aligned}$$

Answer

Verify that $(1-2s) + 2(s) = 1$ ✓

and $2(1-2s) + 4s = 2 - 4s + 4s = 2$ ✓.

Answer

We verify that $(1-2s) + 2(s) = 1$ ✓

and $2(1-2s) + 4(s) = 2 - 4s + 4s = 2$ ✓