Lecture 18

Linear transformations (revisited)

Definition: A linear transformation (between vector spaces)

Let V and W be vector spaces. A function $T : V \to W$ is a **linear transformation** if

- T preserves addition. If v and w are in V, then
$$T(\mathrm{v} + \mathrm{w}) = T(\mathrm{v}) + T(\mathrm{w})$$
- T preserves scalar multiplication. If v is in V and c is in $\mathbb{R}$,
$$T(c\mathrm{v}) = cT(\mathrm{v})$$

They are also called **linear operators**, **linear maps**, or **linear functions**.

If $f(x)$ and $g(x)$ are smooth functions with
$$f'(x) = \sin(x) \text{ and } g'(x) = e^x$$
then the linearity of the derivative operator tells us that
$$\frac{d}{dx}(2f(x) - 4g(x)) = 2\sin(x) - 4e^x$$

Examples of linear transformations

- Many geometric transformations.
  - Reflection, rotation, projection, etc.
- Many differential operators; e.g.

$$H : \mathcal{C}^\infty \to \mathcal{C}^\infty \qquad H(f) := f'' + f$$

- The shift operator on sequences $S : \mathbb{S} \to \mathbb{S}$ which deletes the first term and shifts everything else over.

$$S(x_0, x_1, x_2, x_3, \ldots) := (x_1, x_2, x_3, x_4, \ldots)$$

- Sending a matrix to its transpose defines a linear map

$$\top : \mathbb{R}^{m\times n} \to \mathbb{R}^{n\times m}$$

- Sums, scalar multiples, and compositions of linear transformations are still linear.

The map which sends a 2×2 matrix to its determinant is not a linear transformation. **Why?**

We can use linear transformations to generalize matrix concepts.

Images and kernels of a linear transformation

Let $T : V \to W$ be a linear transformation.

- The **kernel** of T is the subspace of V defined by

$$\ker(T) := \{v \mid v \text{ in } V \text{ such that } T(v) = 0\}$$

- The **image** of T is the subspace of W defined by

$$\mathrm{im}(T) := \{T(v) \mid v \text{ in } V\}$$

Notice that we've snuck in the fact that $\ker(T)$ and $\mathrm{im}(T)$ are subspaces.

Intuition

- The kernel is the set of inputs sent to the zero element by T.
- The image is the set of actual outputs of T.

Exercise 1(a)

Let $F : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by $=$

$$F(p(x)) := (x-2)p'(x)$$

1. Determine whether $x^2 - 1$ is in the kernel of F.
2. Determine whether 16 is in the kernel of F.
3. Determine whether $x^2 - 1$ is in the image of F.
4. Determine whether $x^2 - 4$ is in the image of F.
5. Find the dimension of the kernel of F. Write down a basis for the kernel of F.

One of our favorite theorems generalizes.

The Rank-Nullity Theorem (for linear transformations)

If $T : V \to W$ is a linear transformation, then

$$\dim(\operatorname{im}(T)) + \dim(\ker(T)) = \dim(V)$$

This can be interpreted with the following intuitive picture.

$$\underbrace{(\text{starting dim})}_{\dim(V)} - \underbrace{(\text{dim destroyed by } T)}_{\dim(\ker(T))} = \underbrace{(\text{dim after applying } T)}_{\dim(\operatorname{im}(T))}$$

Exercise 1(b)

1. What is $\dim(\mathrm{im}(F))$ from the previous exercise?
   Solution: Because $\dim(\mathbb{P}_3) = 4$ and we computed $\dim(\ker(F)) = 1$, we know that
   $$\dim(\mathrm{im}(F)) = 4 - 1 = 3$$
2. Write down a basis for $\mathrm{im}(F)$.

Exercise 2

Suppose $T : \mathbb{P}_4 \to \mathbb{P}_6$ is a linear transformation given by

$$T(v) = 0 \text{ for all } v \text{ in } \mathbb{P}_4.$$

What is the dimension of the kernel of T?

Recall that $\mathbb{P}_4$ has dimension 5, and $\mathbb{P}_6$ has dimension 7.

Possible solution A

Since $T(v) = 0$ for all v in $\mathbb{P}_4$, we have $\operatorname{im}(T)) = \{0\}$. So $\dim(\operatorname{im}(T)) = 0$. The theorem tells us

$$\dim(\operatorname{im}(T)) + \dim(\ker(T)) = \dim(\text{domain of } T),$$

$$\text{so} \quad \dim(\operatorname{im}(T)) + \dim(\ker(T)) = \dim(\mathbb{P}_4)$$

$$0 + \dim(\ker(T)) = 5$$

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Suppose $T : \mathbb{P}_4 \to \mathbb{P}_6$ is a linear transformation given by

$$T(v) = 0 \text{ for all } v \text{ in } \mathbb{P}_4.$$

What is the dimension of the kernel of T?

Possible solution B

Since $T(v) = 0$ for all v in $\mathbb{P}_4$, the kernel of T is the entire $\mathbb{P}_4$. So $\ker(T) = \mathbb{P}_4$. So $\dim(\ker(T)) = \dim(\mathbb{P}_4) = 5$.