Lecture 17b

Bases and dimension for vector spaces (part b)

Review

Definition 1: Bases

A subset of a vector space V is called a **basis for V** if every element of V can be written as a linear combination in **exactly one** way.

Definition 3: Dimension

The **dimension** of a vector space is the number of elements in any basis.

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathbb{P}) = \infty$
- $\dim(\mathbb{P}_n) = n + 1$
- $\dim(\mathbb{S}) = \infty$
- $\dim(\mathbb{R}^{m \times n}) = mn$
- $\dim(\mathcal{C}^\infty) = \infty$

Standard bases

$\{e_1, e_2, \ldots, e_n\}$

$\{1, x, x^2, x^3, \ldots\}$

$\{1, x, x^2, x^3, \ldots, x^n\}$

Generalize the relations between spanning sets, linearly independent sets, and bases to **finite**-dimensional vector spaces.

Relations between spanning, linear independence, and bases

Let V be a **finite**-dimensional vector space; i.e. $\dim(V) < \infty$.

- Every spanning set for V contains a basis for V. (direct algorithm)
- Every linearly independent set in V can be extended to a basis for V. (no general algorithm)

Theorem 1 (Dimension and number of vectors)

Let V be a **finite**-dimensional vector space; i.e. $\dim(V) < \infty$.

a. Every basis for V contains exactly $\dim(V)$-many elements.

b. Every spanning set for V contains at least $\dim(V)$-many elements, and equality implies it is a basis.

c. Every linearly independent set in V contains at most $\dim(V)$-many elements, and equality implies it is a basis.

If the dimension is infinite, equality does not imply a basis!

The '2 out of 3 rule' (from Lecture 13b) still holds.

Theorem 2 (The '2 out of 3 Rule' for vector spaces)

Let V be a **finite**-dimensional vector space. Let $v_1, v_2, \ldots, v_k$ be elements in V. If any 2 of the following 3 are true, then the 3rd is automatically true.

- $\{v_1, v_2, \ldots, v_k\}$ is a spanning set for V.
- $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set.
- The dimension of V is k.

So, if any 2 of these are true, then $v_1, v_2, \ldots, v_k$ form a basis for V.

If you know $\dim(V)$ and you want to check if a subset T of V is a basis...

- If T has $\dim(V)$-many vectors, you only need to check one of the two conditions (and one is usually easier).

Theorem 3 (Bounds on dimension)

Let W be a subspace of a finite dimensional subspace V.
(finite-dimensional vector space)

- $0 \leq \dim(W) \leq \dim(V)$.
- If $\dim(W) = 0$, then $W = \{0\}$, i.e. W is the zero subspace.
- If $\dim(W) = \dim(V)$, then $W = V$, i.e. W is all of V.

Exercise 4

Let V be a subspace of $\mathbb{P}_2$ with the following properties.

a. $x^2 - 1$ is in V.

b. $x^2 + 1$ is not in V.

c. V contains a non-zero polynomial $p(x)$ such that $p(3) = 0$.

Find the dimension of V.

Exercise 4

Let V be a subspace of $\mathbb{P}_2$ with the following properties.

a. $x^2 - 1$ is in V.

b. $x^2 + 1$ is not in V.

c. V contains a non-zero polynomial $p(x)$ such that $p(3) = 0$.

Find the dimension of V.

Since V is a subspace of $\mathbb{P}_2$ and $\dim(\mathbb{P}_2) = 3$, the possible dimensions are 0, 1, 2, 3.

$\cancel{0}$ 1 2 $\cancel{3}$

Clue (a) says a non-zero polynomial (x^2-1) is in V, so V is not the zero subspace. So $\dim(V) \neq 0$.

- Clue (b) tells us that there is a polynomial in $\mathbb{P}_2$ which is not in V (the polynomial x^2+1).
- So V is not the entire $\mathbb{P}_2$.
- Thm 3 (Bounds on dimension) on the previous slide tells us that $\dim(V) \neq \dim(\mathbb{P}_2) = 3$.

Claim: $p(x)$ and x^2-1 are linearly independent.

Proof of claim:
- Suppose $\underbrace{c_1\, p(x) + c_2\,(x^2-1)}_{\text{an arbitrary linear combination of } p(x) \text{ and } (x^2-1)} = 0$.

$$c_1\, p(3) + c_2\,(3^2-1) = 0$$

- By clue (c), $0 + c_2\,(8) = 0$
- So $\boxed{c_2 = 0}$
- So $c_1\, p(x) + 0\,(x^2-1) = 0$
- So $c_1\, p(x) = 0$
- Either $c_1 = 0$ or $p(x) = 0$.
- Clue (c) says $p(x) \neq 0$, so $\boxed{c_1 = 0}$

$\therefore$ So $p(x)$ and x^2-1 are linearly independent

Since $\{p(x), x^2-1\}$ is linearly independent, Thm 1 (Dimension & number of vectors) tells us $2 = |\{p(x), x^2-1\}| \leq \dim(V)$.

Since $\dim(V) \neq 3$, $\dim(V) = 2$.

— the end —

Exercise 5

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

In Lec16b, Exercise 5(a), we showed S is subspace of $\mathbb{P}_2$.

a. Find the dimension of S.
b. Find a basis for S.

Exercise 5

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

In Lec16b, Exercise 5(a), we showed S is subspace of $\mathbb{P}_2$.

a. Find the dimension of S.
b. Find a basis for S.

Answer to part (a)

Recall: $\dim(\mathbb{P}_2) = 3$ (since $\{1, x, x^2\}$ is a basis for $\mathbb{P}_2$).

By the "Bounds on dimension", S can have dimension $0, 1, 2, 3$.

~~0~~ 1 2 ~~3~~

S contains $x-5$ which is not the zero element, so $S \neq \{0\}$.

So $\dim(S) \neq 0$.

Not every polynomial in $\mathbb{P}_2$ is in S.

For example, $x^2 - 1$ is not in S.

So $S \neq \mathbb{P}_2$.

So $\dim(S) < \dim(\mathbb{P}_2) = 3$

Note: $x-5$ and $(x-5)^2 = x^2 - 10x + 25$ are both in S.

But $(x-5)$ is not a scalar multiple of $(x-5)^2$, and $(x-5)^2$ is not a scalar multiple of $(x-5)$.

So $\{x-5, (x-5)^2\}$ is linearly independent.

Thm 1 part (c) says: every linearly independent set in S contains at most $\dim(S)$-many elements.

So $\# S \leq \dim(S)$

Since $\dim(S) < 3$, we have

$$2 \leq \dim(S) < 3.$$

So $\boxed{\dim(S) = 2.}$

Exercise 5

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

In Lec16b, Exercise 5(a), we showed S is subspace of $\mathbb{P}_2$.

a. Find the dimension of S.

b. Find a basis for S.

Answer to part (b)

We already computed $\dim(S) = 2$.

We have a subset $\{x-5, (x-5)^2\}$ of S which is linearly independent.

So Let's try to use the '2 out of 3 Rule' for vector spaces.

Since $\{x-5, (x-5)^2\}$ satisfies two out of three properties

(• linearly independent

• having the same number of elements as $\dim(S)$),

we know $\{x-5, (x-5)^2\}$ is a spanning set for S.

Since $\{x-5, (x-5)^2\}$ is linearly independent and is a spanning set for S,

$\{x-5, (x-5)^2\}$ is a basis for S (by definition of basis).

Back to vectors

Bases allow us to return to the world of vectors!

We can use bases to convert elements in general vector spaces into vectors, unlocking the many tools from this class.

Definition 4: The coefficient vector

Let $B := \{v_1, v_2, ..., v_n\}$ be a basis for a vector space V. Given w in V,

$$\text{the } \textbf{coefficient vector} \text{ of } w \text{ in the basis } B \text{ is } \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $c_1 v_2 + c_2 v_2 + \cdots + c_n v_n = w$.

The coefficient vector records the list of coefficients of the unique linear combination equal to w.

E.g. the coefficient vector of $\underbrace{x^2 + 5}_{5 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4}$ in the $\underbrace{\text{standard basis of } \mathbb{P}_4}_{\{1, x, x^2, x^3, x^4\}}$ is ... $\rightarrow \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Back to vectors

A simple but useful observation

Fix a basis for an n-dimensional vector space V. Then **every vector in $\mathbb{R}^n$ is the coefficient vector of a unique element in V.**

We can use this to translate problems in V into problems in $\mathbb{R}^n$.

Back to vectors

Fact 4 (Checking special sets using coefficient vectors)

Fix a basis for an n-dimensional vector space V.

1. A set of elements in V is a spanning set for V if and only if their coefficient vectors is a spanning set for $\mathbb{R}^n$.
2. A set of elements in V is linearly independent in V if and only if their coefficient vectors are linearly independent in $\mathbb{R}^n$.
3. A set of elements in V is a basis if and only if their coefficient vectors form a basis in $\mathbb{R}^n$.

Exercise 6

Let W be the subspace of $\mathcal{C}^\infty$ spanned by $\{e^x, e^{-x}\}$.

a. Show that $\{e^x, e^{-x}\}$ is a basis for W. Show that $\sinh(x), \cosh(x)$ are in W.
b. What is the coefficient vectors of $\sinh(x)$ and $\cosh(x)$ in the basis $\{e^x, e^{-x}\}$?
c. Use the coefficient vectors above to show that $\{\sinh(x), \cosh(x)\}$ is also a basis for W.

Exercise 6

Let W be the subspace of C^∞ spanned by $\{e^x, e^{-x}\}$.

(a) Show that $\{e^x, e^{-x}\}$ is a basis for W. Show that sinh(x) and cosh(x) are in W.

(b) What is the coefficient vectors of $\sinh(x)$ and $\cosh(x)$ in the basis $\{e^x, e^{-x}\}$?

- W is defined to be span$(\{e^x, e^{-x}\})$, so by definition $\{e^x, e^{-x}\}$ is a spanning set for W.
- We showed that $\{e^x, e^{-x}\}$ is linearly independent in Exercise 2 (Lecture 17a).
- So $\{e^x, e^{-x}\}$ is a basis for W.

Recall $\sinh(x) = \dfrac{e^x - e^{-x}}{2} = \frac{1}{2}e^x + \left(-\frac{1}{2}\right)e^{-x}$

and $\cosh(x) = \dfrac{e^x + e^{-x}}{2} = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

So sinh(x) and cosh(x) are in W.

- The coefficient vector of sinh(x) is $\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$.
- The coefficient vector of cosh(x) is $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

Exercise 6

Let W be the subspace of C^∞ spanned by $\{e^x, e^{-x}\}$.

(a) Show that $\{e^x, e^{-x}\}$ is a basis for W. Show that $\sinh(x), \cosh(x)$ are in W.

(b) What is the coefficient vectors of $\sinh(x)$ and $\cosh(x)$ in the basis $\{e^x, e^{-x}\}$?

(c) Use the coefficient vectors above to show that $\{\sinh(x), \cosh(x)\}$ is also a basis for W.

Answer to part (c)

- Since $\{e^x, e^{-x}\}$ is a basis for W, we know $\dim(W) = 2$.
- Fact 4 (3) says: If a set of (two) coefficient vectors is a basis in $\mathbb{R}^2$, then the corresponding elements form a basis for W.
- To check that $\left\{\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$, we just need to verify that $\left\{\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\right\}$ is linearly independent (by the '2 out of 3' rule, since we know $\dim(\mathbb{R}^2) = 2$):
- Need to show $\underbrace{\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}}_{\text{concatenation}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has one unique solution $(c_1 = c_2 = 0)$.

Row reduce the augmented matrix:

$$\left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \xrightarrow[R_2 \mapsto 2R_2]{R_1 \mapsto 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto \frac{1}{2}R_2]{} \underbrace{\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]}_{\text{in REF}}$$

Every column left of the vertical line has a leading 1.

This shows that $(*)$ has one unique solution, the trivial solution $c_1 = c_2 = 0$.

- So $\left\{\begin{bmatrix} \frac{1}{2} \\ -1/2 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right\}$ is linearly independent, therefore it's a basis for $\mathbb{R}^2$ (by the '2 out of 3' rule).

- Since the coefficient vectors of $\sinh(x)$ and $\cosh(x)$ form a basis for $\mathbb{R}^2$, we know $\{\sinh(x), \cosh(x)\}$ is a basis for W (due to $\dim(W) = 2$).