Lecture 17a

Bases and dimension for vector spaces (part a)

Review

Recall: Vector spaces

A **vector space** is a set V in which

- we know how to add any two elements v, w in V, and
- we know how to multiply any v in V by any scalar r in $\mathbb{R}$,

which obey some axioms (the essential properties of arithmetic).

Examples of vector spaces

- For any n: $\mathbb{R}^n$, the set of vectors of height n.
- $\mathbb{P}$, the set of polynomials in x.
- For any n: $\mathbb{P}_n$, the set of polynomials in x of degree at most n.
- $\mathbb{S}$, the set of sequences.
- For any m and n: $\mathbb{R}^{m \times n}$, the set of $m \times n$-matrices.
- $\mathcal{C}^\infty$, the set of smooth functions of x.
- **Any subspace of a vector space is a vector space.**

Review

Recall three definitions involving linear combinations.

Recall: Goldilocks and the three properties

A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a subspace V of $\mathbb{R}^n$ is...

1. ...a **spanning set for V** if every element of V can be written as a linear combination in at least one way,
2. ...a **linearly independent set** if every element of V can be written as a linear combination in at most one way, and
3. ...a **basis for V** if every element of V can be written as a linear combination in exactly one way.

Linear combinations make sense in any vector space!

Definition 1: Goldilocks and the three properties, generalized

A set of elements $\{v_1, v_2, \dots, v_k\}$ in a vector space V is...

a. ...a **spanning set for V** if every element of V can be written as a linear combination in at least one way,
b. ...a **linearly independent set** if every element of V can be written as a linear combination in at most one way, and
c. ...a **basis for V** if every element of V can be written as a linear combination in exactly one way.

To check these properties, some of our tools generalize...

Definition 1(b): Checking linear independence

A set of elements $\{v_1, v_2, ..., v_n\}$ in a vector space V is linearly independent if the only linear combination which is equal to the zero element is the trivial linear combination. That is, if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

then each of $c_1, c_2, ..., c_n$ must be 0.

...but not all our tools generalize.

Don't try to concatenate!

We can no longer turn a linear combination into multiplication by the concatenated matrix.

Exercise 1(a)

- Determine whether $\{x, x+1, x+2\}$ is a spanning set for the vector space $\mathbb{P}_1$.

Exercise 1(b)

- Determine whether $\{x-1, x^2-1, x^2-x\}$ is linearly independent in the vector space $\mathbb{P}$.

Exercise 1(c)

- Determine whether $\{x^2, (x-1)^2, (x-2)^2\}$ is a basis for the vector space $\mathbb{P}_2$.

Exercise 2

Show that $\{e^x, e^{-x}\}$ is linearly independent in the vector space $\mathcal{C}^\infty$.

Exercise 1(a)

- Determine whether $\underbrace{\{x, x+1, x+2\}}_{S:=}$ is a spanning set for the vector space $\mathbb{P}_1$.

- Recall that $\mathbb{P}_1 = \{\text{polynomials of degree at most } 1\}$
$= \{b_1 x + b_0 \text{ for some } b_1, b_0 \text{ in } \mathbb{R}\}$

- We want to check:
Set $b_1 x + b_0 = \underbrace{c_1 x + c_2(x+1) + c_3(x+2)}_{\text{a linear combination of } S}$

Can we find at least one solution (for c_1, c_2, c_3) for all b_1, b_0?

- Rewrite the RHS
$b_1 x + b_0 = (c_1 + c_2 + c_3)\, x + (1.c_2 + 2.c_3)$

This is equivalent to

$$\left.\begin{aligned} b_1 &= c_1 + c_2 + c_3 \\ b_0 &= \phantom{c_1 + {}} c_2 + 2c_3 \end{aligned}\right\} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}$$

This is a system of linear equations, with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & b_0 \end{array}\right]$$ which is already in REF.

The right-most column has no leading 1, so the system is consistent!

So there exist c_1, c_2, c_3 in $\mathbb{R}$ where $b_1 x + b_0 = c_1 x + c_2(x+1) + c_3(x+2)$.

Therefore, $\boxed{S = \{x, x+1, x+2\} \text{ is a spanning set for } \mathbb{P}_1.}$

Exercise 1(b)

- Determine whether $\{x-1, x^2-1, x^2-x\}$ is linearly independent in the vector space $\mathbb{P}$. $T :=$

Set $c_1(x-1) + c_2(x^2-1) + c_3(x^2-x) = 0$.

Check: Are there solutions other than $c_1 = c_2 = c_3 = 0$?

If no, then T is linearly independent.

If yes, then T is $\underbrace{\text{not linearly independent}}_{\text{(linearly dependent)}}$

Rewrite $c_1(x-1) + c_2(x^2-1) + c_3(x^2-x) = 0$.

$$(c_2 + c_3)x^2 + (c_1 - c_3)x + (-c_1 - c_2) = 0 = 0x^2 + 0x + 0$$

$$\left.\begin{array}{rcl} c_2 + c_3 &=& 0 \\ c_1 \quad - c_3 &=& 0 \\ -c_1 - c_2 \quad &=& 0 \end{array}\right\} \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto R_1 + R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto R_2 + R_3]{} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{in REF}}$$

Not every column left of the vertical line has a leading 1.

The system has infinitely many solutions. In particular, there are solutions other than $c_1 = c_2 = c_3 = 0$.

So $\boxed{T = \{x-1, x^2-1, x^2-x\} \text{ is } \underline{\underline{\text{not}}} \text{ linearly independent.}}$

Exercise 1(c)

- Determine whether $\{x^2, (x-1)^2, (x-2)^2\}$ is a basis for the vector space $\mathbb{P}_2$. $K :=$

Recall $\mathbb{P}_2 = \{$ polynomials of degree at most $2\}$.

$= \{ b_2 x^2 + b_1 x + b_0$ for b_2, b_1, b_0 in $\mathbb{R}\}$.

Set $b_2 x^2 + b_1 x + b_0 = \underbrace{c_1 x^2 + c_2 (x-1)^2 + c_3 (x-2)^2}_{\text{a linear combination of } K}$

Check whether this has 0, 1, or infinitely many solutions.
If 0, this means K is not spanning
If 1, this means K is $\underbrace{\text{spanning and linearly independent}}_{\text{(a basis)}}$
If infinitely many, this means K is spanning but K is not linearly independent.

$$b_2 x^2 + b_1 x + b_0 = c_1 x^2 + c_2(x^2 - 2x + 1) + c_3(x^2 - 4x + 4)$$

$$b_2 x^2 + b_1 x + b_0 = (c_1 + c_2 + c_3)x^2 + (-2c_2 - 4c_3)x + (c_2 + 4c_3)$$

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= b_2 \\ -2c_2 - 4c_3 &= b_1 \\ c_2 + 4c_3 &= b_0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

Then write as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & -2 & -4 & b_1 \\ 0 & 1 & 4 & b_0 \end{array}\right] \xrightarrow[R_2 \mapsto -\frac{1}{2}R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 1 & 2 & -\frac{b_1}{2} \\ 0 & 1 & 4 & b_0 \end{array}\right] \xrightarrow[R_3 \mapsto -R_2 + R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 1 & 2 & -\frac{b_1}{2} \\ 0 & 0 & 2 & b_0 + \frac{b_1}{2} \end{array}\right] \xrightarrow[R_3 \mapsto \frac{1}{2}R_3]{} \underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 1 & 2 & -\frac{b_1}{2} \\ 0 & 0 & 1 & \frac{b_0}{2} + \frac{b_1}{4} \end{array}\right]}_{\text{in REF}}$$

- The right column has no leading 1, so the system is consistent
- Every column to the left of the vertical line has a leading 1, so there is one unique solution.

$\therefore$ The equation

$$b_2 x^2 + b_1 x + b_0 = (c_1 + c_2 + c_3)x^2 + (-2c_2 - 4c_3)x + (c_2 + 4c_3)$$

has one unique solution (c_1, c_2, c_3) for each b_2, b_1, b_0 in $\mathbb{R}$.

$\therefore$ Every element in $\mathbb{P}_2$ can be written as a linear combination of K in exactly one way.

$\therefore$ $\boxed{K \text{ is a basis for } \mathbb{P}_2}$

Exercise 2

Show that $\{e^x, e^{-x}\}$ is linearly independent in the vector space C^∞.

Set $c_1 e^x + c_2 e^{-x} = 0$.

We want to show that the only solution is $c_1 = 0$, $c_2 = 0$.

Try plugging in values of x to get equations.

<u>Plug in $x=0$</u>

$$c_1 e^0 + c_2 e^{-0} = 0$$
$$c_1 + c_2 = 0$$

<u>Plug in $x=1$</u>

$$c_1 e^1 + c_2 e^{-1} = 0$$
$$c_1 e^2 + c_2 = 0$$
$$c_1 e^2 - c_1 = 0$$
$$c_1(e^2 - 1) = 0$$

I know $e^2 \neq 1$, so $e^2 - 1 \neq 0$.

So $c_1 = 0$.

Therefore $c_2 = 0$.

We've shown $c_1 e^x + c_2 e^{-x} = 0$ implies $c_1 = c_2 = 0$.

$\therefore$ $\boxed{\{e^x, e^{-x}\} \text{ is linearly independent.}}$

— the end —

Definition 2: Linear combinations of an infinite set

We can't add infinitely many things, so a **linear combination** of an infinite set is defined as a linear combination of any finite subset.

Alternatively, it's a linear combination of the whole set with finitely-many non-zero coefficients.

Exercise 3

Show that the set of powers of x

$$\{1, x, x^2, x^3, x^4, \ldots\}$$

is a basis for $\mathbb{P}$.

Exercise 3

Show that the set of powers of x

$$B := \{1, x, x^2, x^3, x^4, \ldots\}$$

is a basis for $\mathbb{P}$.

- To show that B is a spanning set of $\mathbb{P}$, let $f(x)$ be in $\mathbb{P}$.
  Then $f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_2 \cdot x^2 + a_1 \cdot x + a_0 \cdot 1$
  for some $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ in $\mathbb{R}$.
  But this shows that $f(x)$ is a linear combination of B.
  So B is a spanning set of $\mathbb{P}$.

- To show that B is linearly independent in $\mathbb{P}$, take a linear combination of B and set it to 0:

$$b_0 1 + b_1 x + b_2 x^2 + \ldots + b_m x^m = 0$$

  The only solution to this equation is $b_0 = b_1 = \ldots = b_m = 0$,
  so B is linearly independent.

So $\{1, x, x^2, x^3, \ldots\}$ is a basis for $\mathbb{P}$ (called the standard basis for $\mathbb{P}$).

Many of our favorite vector spaces have simple, ‘standard’ bases.

Standard bases for some vector spaces

- The standard basis vectors $\{e_1, e_2, e_3, \ldots, e_n\}$ form a basis for $\mathbb{R}^n$. E.g. $\mathbb{R}^3$ has standard basis $\left\{ e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.
- The $\underbrace{\text{powers of } x}_{\{1, x, x^2, x^3, x^4, \ldots\}}$ form a basis for $\mathbb{P}$.
- The powers of x less or equal to n form a basis for $\mathbb{P}_n$.
  E.g. $\mathbb{P}_4$ has standard basis $\underbrace{\{1, x, x^2, x^3, x^4\}}_{\text{five elements!}}$
- The matrices which are 1 in one entry and 0 elsewhere form a basis for $\mathbb{R}^{m \times n}$.
  E.g. $\mathbb{R}^{2\times 2}$ has standard basis $\left\{ \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\}$

Not every vector space has a ‘standard’ basis.

We have a generalization of one of our big theorems.

Theorem (The Invariance Theorem)

Every basis for a vector space have the same number of elements.

Definition 3: Dimension

The **dimension** of a vector space is the number of elements in any basis.

Intuitively, the dimension is the smallest amount of numbers you need to describe an arbitrary element in the vector space.

Examples

E.g. For $\mathbb{R}^5$, you need 5 numbers to describe $\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix}$.

- $\dim(\mathbb{R}^n) = n$.
- $\dim(\mathbb{P}_n) = n + 1$. This throws people off!
  (remember that $\{1, x, x^2, ..., x^n\}$ has $(n + 1)$-many elements)

E.g. For $\mathbb{P}_2$, an arbitrary element looks like $ax^2 + bx + c$.

Three numbers a, b, c.

- $\dim(\mathbb{R}^{m \times n}) = mn$.

E.g. For $\mathbb{R}^{2\times 3}$, an arbitrary element looks like $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.

Infinite dimensional vector spaces

The dimension of a vector space can be **infinite**!

If there is no finite basis for V, we say that $\dim(V) = \infty$.

Examples

- $\dim(\mathbb{P}) = \infty$, because the standard basis is infinite:

$$\{1, x, x^2, x^3, ...\}$$

- $\dim(\mathbb{S}) = \infty$.
- $\dim(\mathcal{C}^\infty) = \infty$.

Intuitively, this says there is no way to describe every polynomial, sequence, or smooth function using n-many numbers, for fixed n.