Week 14/15 Individual Worksheet (Six problems total, based on Lecture 14 to 17) **Math 3333**

- Due Monday, December 7, 2020 at noon on Gradescope
- Write on your own paper. Put each question on its own page.
- If you have a printer or a tablet/stylus, you may write directly on the worksheet.
- Submit individually (on Gradescope). You are encouraged to work with your classmates and also ask for help during class/office hours.

1 Lecture 14a: Basis algorithm for the kernel of a matrix

$$\text{Let} \quad A := \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

The number -2 is one of the eigenvalues of A. Let W be the -2-eigenspace of A.

(i) Find a basis for W. (See the end of Lecture 14a for an example)

(ii) What is the dimension of W?

2 Lecture 15b: Finding eigenbases

$$\text{Let } A := \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}.$$

a.) Without actually finding an eigenbasis, use one of the theorems in Lecture egunawan.github.io/notes/la/lecture15b.pdf 15b to show that A has an eigenbasis.

b.) OK, now find an eigenbasis of A. (Please write down the corresponding eigenvalue for each vector in your eigenbasis. You'll need it to do the next problem.)

Optional sanity check: Since an eigenbasis of A is a basis for $\mathbb{R}^2$, a concatenation of your eigenbasis should have determinant non-zero.

Optional sanity check: Since each vector in an eigenbasis of A is an eigenvector of A, it must be that $Av = cv$ for each vector v where c is the corresponding eigenvalue.

3 Lecture 15c

$$\text{Let } A := \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}, \text{ from the previous question.}$$

a. Use the eigenbasis of A you found in the previous question to write $A = BDB^{-1}$ **where D** $:= \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.

Then check your work by multiplying out your factorization.

b. Compute A^{100} (See the solutions in Lecture 15c for guidance).

4 Lecture 16a: Vector spaces

Recall that $\mathbb{P}_2$ denotes the set of polynomials in x of degree at most 2. Show that every polynomial in $\mathbb{P}_2$ can be written as a linear combination of the following three polynomials.

$$x^2 + x + 1, \qquad x^2 + 2x + 1, \qquad x^2 + 4x + 4$$

(Hint: Start by writing

'Let $f(x) = b_2x^2 + b_1x + b_0$ for some numbers b_2, b_1, b_0. We need to show that there are a, b, c in $\mathbb{R}$ such that

$$f(x) = a(x^2 + x + 1) + b(x^2 + 2x + 1) + c(x^2 + 4x + 4).$$
$$b_2x^2 + b_1x + b_0 = a(x^2 + x + 1) + b(x^2 + 2x + 1) + c(x^2 + 4x + 4)$$ '

Then start computing like in the solution to Exercise 1(c) of Lec 16a. See also Exercise 1(a) in egunawan.github.io/la/notes/lecture17a.pdf.)

Optional sanity check (or partial credit): Write a polynomial in $\mathbb{P}_2$ (for example, x or 0 or $x^2 + 1$) as a linear combination of the three polynomials

$$x^2 + x + 1, \qquad x^2 + 2x + 1, \qquad x^2 + 4x + 4.$$

5 Lecture 16b: Vector spaces (subspaces)

Recall that $\mathbb{P}$ denotes the vector space of all polynomials in x.

a. Determine whether $V := \{f(x) \text{ in } \mathbb{P} \mid f(1) = 0\}$ is or is not a subspace of $\mathbb{P}$. Prove your answer. (Hint: Your submission should look like the "SAMPLE STUDENT ANSWER" templates in egunawan.github.io/la/notes/lecture16b.pdf.)

b. Determine whether $W := \{f(x) \text{ in } \mathbb{P} \mid f(1) = 2\}$ is or is not a subspace of $\mathbb{P}$. Prove your answer. (Hint: Your submission should look like the "SAMPLE STUDENT ANSWER" templates in lecture 16b.)

6 Lecture 17: Bases and dimensions for vector spaces

Let S denote the set of smooth functions $f(x)$ in $\mathcal{C}^\infty$ such that $f''(x) = -f(x)$. That is,

$$S = \{f(x) \text{ in } \mathcal{C}^\infty \mid f''(x) = -f(x)\}.$$

Lec 16b Exercise 4 showed that S is a subspace of $\mathcal{C}^\infty$ (so S is a vector space). We also showed that $\sin(x)$ is in S.

a. Is $\cos(x)$ in S?

b. Show that the set $\{\sin(x), \cos(x)\}$ is a linearly independent set.

(Hint1: Set an arbitrary linear combination to 0, then show the coefficients must be 0 by plugging in convenient values, like in Exercise 2 of Lec 17a. Hint2: what number r would give $\cos(r) = 0$?)

c. Let's say you know $\dim(S) = 2$ (You can take this for granted.) Determine whether the set $\{\sin(x), \cos(x)\}$ is a basis for the vector space S. Give a detailed explanation in complete sentences.

(Hint: Use the '2 out of 3' rule for vector spaces (Theorem 2 in egunawan.github.io/la/notes/lecture17b.pdf)

d. Use other parts of this question to either find every function $f(x)$ such that $f''(x) = -f(x)$, $f(0) = 5$, and $f'(0) = 7$ or show that no such function can exist.