Selected Exercise Answers

Section 1.1

1.1.1 b. $2(2s+12t+13)+5s+9(-s-3t-3)+3t=-1$; $(2s+12t+13)+2s+4(-s-3t-3)=1$

1.1.2 b. $x=t$, $y=\frac{1}{3}(1-2t)$ or $x=\frac{1}{2}(1-3s)$, $y=s$

d. $x=1+2s-5t$, $y=s$, $z=t$ or $x=s$, $y=t$, $z=\frac{1}{5}(1-s+2t)$

1.1.4 $x=\frac{1}{4}(3+2s)$, $y=s$, $z=t$

1.1.5 a. No solution if $b\neq 0$. If $b=0$, *any* x is a solution.

b. $x=\frac{b}{a}$

1.1.7 b. $\left[\begin{array}{rr|r} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$

d. $\left[\begin{array}{rrr|r} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 2 \end{array}\right]$

1.1.8 b.

$$\begin{array}{rcr} 2x-y & = & -1 \\ -3x+2y+z & = & 0 \\ y+z & = & 3 \end{array}$$

or

$$\begin{array}{rcr} 2x_1-x_2 & = & -1 \\ -3x_1+2x_2+x_3 & = & 0 \\ x_2+x_3 & = & 3 \end{array}$$

1.1.9 b. $x=-3$, $y=2$

d. $x=-17$, $y=13$

1.1.10 b. $x=\frac{1}{9}$, $y=\frac{10}{9}$, $z=-\frac{7}{3}$

1.1.11 b. No solution

1.1.14 b. F. $x+y=0$, $x-y=0$ has a unique solution.

d. T. Theorem 1.1.1.

1.1.16 $x'=5$, $y'=1$, so $x=23$, $y=-32$

1.1.17 $a=-\frac{1}{9}$, $b=-\frac{5}{9}$, $c=\frac{11}{9}$

1.1.19 \$4.50, \$5.20

Section 1.2

1.2.1 b. No, no

d. No, yes

f. No, no

1.2.2 b. $\left[\begin{array}{rrrrrrr} 0 & 1 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$

1.2.3 b. $x_1=2r-2s-t+1$, $x_2=r$, $x_3=-5s+3t-1$, $x_4=s$, $x_5=-6t+1$, $x_6=t$

d. $x_1=-4s-5t-4$, $x_2=-2s+t-2$, $x_3=s$, $x_4=1$, $x_5=t$

1.2.4 b. $x=-\frac{1}{7}$, $y=-\frac{3}{7}$

d. $x=\frac{1}{3}(t+2)$, $y=t$

f. No solution

1.2.5 b. $x=-15t-21$, $y=-11t-17$, $z=t$

d. No solution

f. $x=-7$, $y=-9$, $z=1$

h. $x=4$, $y=3+2t$, $z=t$

1.2.6 b. Denote the equations as E_1, E_2, and E_3. Apply gaussian elimination to column 1 of the augmented matrix, and observe that $E_3-E_1=-4(E_2-E_1)$. Hence $E_3=5E_1-4E_2$.

1.2.7 b. $x_1=0$, $x_2=-t$, $x_3=0$, $x_4=t$

d. $x_1=1$, $x_2=1-t$, $x_3=1+t$, $x_4=t$

1.2.8 b. If $ab\neq 2$, unique solution $x=\frac{-2-5b}{2-ab}$, $y=\frac{a+5}{2-ab}$. If $ab=2$: no solution if $a\neq -5$; if $a=-5$, the solutions are $x=-1+\frac{2}{5}t$, $y=t$.

d. If $a\neq 2$, unique solution $x=\frac{1-b}{a-2}$, $y=\frac{ab-2}{a-2}$. If $a=2$, no solution if $b\neq 1$; if $b=1$, the solutions are $x=\frac{1}{2}(1-t)$, $y=t$.

1.2.9 b. Unique solution $x = -2a + b + 5c$, $y = 3a - b - 6c$, $z = -2a + b + c$, for any a, b, c.

d. If $abc \neq -1$, unique solution $x = y = z = 0$; if $abc = -1$ the solutions are $x = abt$, $y = -bt$, $z = t$.

f. If $a = 1$, solutions $x = -t$, $y = t$, $z = -1$. If $a = 0$, there is no solution. If $a \neq 1$ and $a \neq 0$, unique solution $x = \frac{a-1}{a}$, $y = 0$, $z = \frac{-1}{a}$.

1.2.10 b. 1

d. 3

f. 1

1.2.11 b. 2

d. 3

f. 2 if $a = 0$ or $a = 2$; 3, otherwise.

1.2.12 b. False. $A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

d. False. $A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

f. False. $\begin{array}{r} 2x - y = 0 \\ -4x + 2y = 0 \end{array}$ is consistent but $\begin{array}{r} 2x - y = 1 \\ -4x + 2y = 1 \end{array}$ is not.

h. True, A has 3 rows, so there are at most 3 leading 1s.

1.2.14 b. Since one of $b - a$ and $c - a$ is nonzero, then

$$\left[\begin{array}{ccc} 1 & a & b+c \\ 1 & b & c+a \\ 1 & b & c+a \end{array}\right] \to \left[\begin{array}{ccc} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{array}\right] \to$$
$$\left[\begin{array}{ccc} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 0 & b+c+a \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

1.2.16 b. $x^2 + y^2 - 2x + 6y - 6 = 0$

1.2.18 $\frac{5}{20}$ in A, $\frac{7}{20}$ in B, $\frac{8}{20}$ in C.

Section 1.3

1.3.1 b. False. $A = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

d. False. $A = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$

f. False. $A = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

h. False. $A = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

1.3.2 b. $a = -3$, $x = 9t$, $y = -5t$, $z = t$

d. $a = 1$, $x = -t$, $y = t$, $z = 0$; or $a = -1$, $x = t$, $y = 0$, $z = t$

1.3.3 b. Not a linear combination.

d. $\mathbf{v} = \mathbf{x} + 2\mathbf{y} - \mathbf{z}$

1.3.4 b. $\mathbf{y} = 2\mathbf{a}_1 - \mathbf{a}_2 + 4\mathbf{a}_3$.

1.3.5 b. $r \left[\begin{array}{c} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right] + s \left[\begin{array}{c} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{array}\right] + t \left[\begin{array}{c} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{array}\right]$

d. $s \left[\begin{array}{c} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{array}\right] + t \left[\begin{array}{c} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{array}\right]$

1.3.6 b. The system in (a) has nontrivial solutions.

1.3.7 b. By Theorem 1.2.2, there are $n - r = 6 - 1 = 5$ parameters and thus infinitely many solutions.

d. If R is the row-echelon form of A, then R has a row of zeros and 4 rows in all. Hence R has $r = \text{rank } A = 1$, 2, or 3. Thus there are $n - r = 6 - r = 5$, 4, or 3 parameters and thus infinitely many solutions.

1.3.9 b. That the graph of $ax + by + cz = d$ contains three points leads to 3 linear equations homogeneous in variables a, b, c, and d. Apply Theorem 1.3.1.

1.3.11 There are $n - r$ parameters (Theorem 1.2.2), so there are nontrivial solutions if and only if $n - r > 0$.

Section 1.4

1.4.1 b.
$$\begin{aligned} f_1 &= 85 - f_4 - f_7 \\ f_2 &= 60 - f_4 - f_7 \\ f_3 &= -75 + f_4 + f_6 \\ f_5 &= 40 - f_6 - f_7 \end{aligned}$$
f_4, f_6, f_7 parameters

1.4.2 b. $f_5 = 15$
$25 \leq f_4 \leq 30$

1.4.3 b. CD

Section 1.5

1.5.2 $I_1 = -\frac{1}{5}, I_2 = \frac{3}{5}, I_3 = \frac{4}{5}$

1.5.4 $I_1 = 2, I_2 = 1, I_3 = \frac{1}{2}, I_4 = \frac{3}{2}, I_5 = \frac{3}{2}, I_6 = \frac{1}{2}$

Section 1.6

1.6.2 $2NH_3 + 3CuO \to N_2 + 3Cu + 3H_2O$

1.6.4 $15Pb(N_3)_2 + 44Cr(MnO_4)_2 \to 22Cr_2O_3 + 88MnO_2 + 5Pb_3O_4 + 90NO$

Supplementary Exercises for Chapter 1

Supplementary Exercise 1.1. b. No. If the corresponding planes are parallel and distinct, there is no solution. Otherwise they either coincide or have a whole common line of solutions, that is, at least one parameter.

Supplementary Exercise 1.2. b. $x_1 = \frac{1}{10}(-6s - 6t + 16), x_2 = \frac{1}{10}(4s - t + 1), x_3 = s, x_4 = t$

Supplementary Exercise b.. b. If $a = 1$, no solution. If $a = 2, x = 2 - 2t, y = -t, z = t$. If $a \neq 1$ and $a \neq 2$, the unique solution is $x = \frac{8-5a}{3(a-1)}, y = \frac{-2-a}{3(a-1)}, z = \frac{a+2}{3}$

Supplementary Exercise 1.4. $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \to \begin{bmatrix} R_1 + R_2 \\ R_2 \end{bmatrix} \to \begin{bmatrix} R_1 + R_2 \\ -R_1 \end{bmatrix} \to \begin{bmatrix} R_2 \\ -R_1 \end{bmatrix} \to \begin{bmatrix} R_2 \\ R_1 \end{bmatrix}$

Supplementary Exercise 1.6. $a = 1, b = 2, c = -1$

Supplementary Exercise 1.8. The (real) solution is $x = 2, y = 3 - t, z = t$ where t is a parameter. The given complex solution occurs when $t = 3 - i$ is complex. If the real system has a unique solution, that solution is real because the coefficients and constants are all real.

Supplementary Exercise 1.9. b. 5 of brand 1, 0 of brand 2, 3 of brand 3

Section 2.1

2.1.1 b. $(a\ b\ c\ d) = (-2, -4, -6, 0) + t(1, 1, 1, 1)$, t arbitrary

d. $a = b = c = d = t$, t arbitrary

2.1.2 b. $\begin{bmatrix} -14 \\ -20 \end{bmatrix}$

d. $(-12, 4, -12)$

f. $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 4 & -1 \\ -1 & -6 \end{bmatrix}$

2.1.3 b. $\begin{bmatrix} 15 & -5 \\ 10 & 0 \end{bmatrix}$

d. Impossible

f. $\begin{bmatrix} 5 & 2 \\ 0 & -1 \end{bmatrix}$

h. Impossible

2.1.4 b. $\begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$

2.1.5 b. $A = -\frac{11}{3}B$

2.1.6 b. $X = 4A - 3B, Y = 4B - 5A$

2.1.7 b. $Y = (s, t), X = \frac{1}{2}(1 + 5s, 2 + 5t)$; s and t arbitrary

2.1.8 b. $20A - 7B + 2C$

2.1.9 b. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $(p, q, r, s) = \frac{1}{2}(2d, a + b - c - d, a - b + c - d, -a + b + c + d)$.

2.1.11 b. If $A + A' = 0$ then $-A = -A + 0 = -A + (A + A') = (-A + A) + A' = 0 + A' = A'$

2.1.13 b. Write $A = \text{diag}(a_1, \dots, a_n)$, where $a_1, \dots, a_n$ are the main diagonal entries. If $B = \text{diag}(b_1, \dots, b_n)$ then $kA = \text{diag}(ka_1, \dots, ka_n)$.

2.1.14 b. $s = 1$ or $t = 0$

d. $s = 0$, and $t = 3$

2.1.15 b. $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 7 \\ -\frac{9}{2} & -5 \end{bmatrix}$

2.1.16 b. $A = A^T$, so using Theorem 2.1.2, $(kA)^T = kA^T = kA$.

2.1.19 b. False. Take $B = -A$ for any $A \neq 0$.

d. True. Transposing fixes the main diagonal.

f. True.
$(kA+mB)^T = (kA)^T + (mB)^T = kA^T + mB^T = kA + mB$

2.1.20 c. Suppose $A = S + W$, where $S = S^T$ and $W = -W^T$. Then $A^T = S^T + W^T = S - W$, so $A + A^T = 2S$ and $A - A^T = 2W$. Hence $S = \frac{1}{2}(A + A^T)$ and $W = \frac{1}{2}(A - A^T)$ are uniquely determined by A.

2.1.22 b. If $A = [a_{ij}]$ then $(kp)A = [(kp)a_{ij}] = [k(pa_{ij})] = k[pa_{ij}] = k(pA)$.

Section 2.2

2.2.1 b.

$$\begin{aligned} x_1 - 3x_2 - 3x_3 + 3x_4 &= 5 \\ 8x_2 \quad\quad + 2x_4 &= 1 \\ x_1 + 2x_2 + 2x_3 \quad\quad &= 2 \\ x_2 + 2x_3 - 5x_4 &= 0 \end{aligned}$$

2.2.2
$$x_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ -2 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 7 \\ 9 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 8 \\ 12 \end{bmatrix}$$

2.2.3 b.
$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ -4x_2 + 5x_3 \end{bmatrix}$$

d.
$$A\mathbf{x} = \begin{bmatrix} 3 & -4 & 1 & 6 \\ 0 & 2 & 1 & 5 \\ -8 & 7 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 0 \\ -8 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3x_1 - 4x_2 + x_3 + 6x_4 \\ 2x_2 + x_3 + 5x_4 \\ -8x_1 + 7x_2 - 3x_3 \end{bmatrix}$$

2.2.4 b. To solve $A\mathbf{x} = \mathbf{b}$ the reduction is
$$\left[\begin{array}{rrrr|r} 1 & 3 & 2 & 0 & 4 \\ 1 & 0 & -1 & -3 & 1 \\ -1 & 2 & 3 & 5 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & -1 & -3 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
so the general solution is
$$\begin{bmatrix} 1+s+3t \\ 1-s-t \\ s \\ t \end{bmatrix}.$$

Hence $(1+s+3t)\mathbf{a}_1 + (1-s-t)\mathbf{a}_2 + s\mathbf{a}_3 + t\mathbf{a}_4 = \mathbf{b}$ for any choice of s and t. If $s = t = 0$, we get $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{b}$; if $s = 1$ and $t = 0$, we have $2\mathbf{a}_1 + \mathbf{a}_3 = \mathbf{b}$.

2.2.5 b.
$$\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

d.
$$\begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

2.2.6 We have $A\mathbf{x}_0 = \mathbf{0}$ and $A\mathbf{x}_1 = \mathbf{0}$ and so $A(s\mathbf{x}_0 + t\mathbf{x}_1) = s(A\mathbf{x}_0) + t(A\mathbf{x}_1) = s \cdot \mathbf{0} + t \cdot \mathbf{0} = \mathbf{0}$.

2.2.8 b.
$$\mathbf{x} = \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \left(s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right).$$

2.2.10 b. False. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

d. True. The linear combination $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$ equals $A\mathbf{x}$ where $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ by Theorem 2.2.1.

f. False. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, then

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \neq s \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for any } s \text{ and } t.$$

h. False. If $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, there is a solution for $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ but not for $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2.2.11 b. Here $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

d. Here $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

2.2.13 b. Here

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so the matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.2.16 Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ in terms of its columns. If $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$ where the x_i are scalars, then $A\mathbf{x} = \mathbf{b}$ by Theorem 2.2.1 where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$. That is, $\mathbf{x}$ is a solution to the system $A\mathbf{x} = \mathbf{b}$.

2.2.18 b. By Theorem 2.2.3, $A(t\mathbf{x}_1) = t(A\mathbf{x}_1) = t \cdot \mathbf{0} = \mathbf{0}$; that is, $t\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{0}$.

2.2.22 If A is $m \times n$ and $\mathbf{x}$ and $\mathbf{y}$ are n-vectors, we must show that $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$, and write $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$. Then $\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 & x_2+y_2 & \cdots & x_n+y_n \end{bmatrix}^T$, so Definition 2.1 and Theorem 2.1.1 give $A(\mathbf{x}+\mathbf{y}) = (x_1+y_1)\mathbf{a}_1 + (x_2+y_2)\mathbf{a}_2 + \cdots + (x_n+y_n)\mathbf{a}_n = (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n) + (y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \cdots + y_n\mathbf{a}_n) = A\mathbf{x} + A\mathbf{y}$.

Section 2.3

2.3.1 b. $\begin{bmatrix} -1 & -6 & -2 \\ 0 & 6 & 10 \end{bmatrix}$

d. $\begin{bmatrix} -3 & -15 \end{bmatrix}$

f. $[-23]$

h. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

j. $\begin{bmatrix} aa' & 0 & 0 \\ 0 & bb' & 0 \\ 0 & 0 & cc' \end{bmatrix}$

2.3.2 b. $BA = \begin{bmatrix} -1 & 4 & -10 \\ 1 & 2 & 4 \end{bmatrix}$, $B^2 = \begin{bmatrix} 7 & -6 \\ -1 & 6 \end{bmatrix}$,

$CB = \begin{bmatrix} -2 & 12 \\ 2 & -6 \\ 1 & 6 \end{bmatrix}$

$AC = \begin{bmatrix} 4 & 10 \\ -2 & -1 \end{bmatrix}$, $CA = \begin{bmatrix} 2 & 4 & 8 \\ -1 & -1 & -5 \\ 1 & 4 & 2 \end{bmatrix}$

2.3.3 b. $(a, b, a_1, b_1) = (3, 0, 1, 2)$

2.3.4 b. $A^2 - A - 6I =$
$\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.3.5 b. $A(BC) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -9 & -16 \\ 5 & 1 \end{bmatrix} =$
$\begin{bmatrix} -14 & -17 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 5 & 8 \end{bmatrix} =$
$(AB)C$

2.3.6 b. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, compare entries an AE and EA.

2.3.7 b. $m \times n$ and $n \times m$ for some m and n

2.3.8 b. i. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

2.3.12 b. $A^{2k} = \left[\begin{array}{cc|cc} 1 & -2k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ for $k = 0, 1, 2, \dots$,

$A^{2k+1} = A^{2k}A = \left[\begin{array}{cc|cc} 1 & -(2k+1) & 2 & -1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$ for $k = 0, 1, 2, \dots$

2.3.13 b. $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I_{2k}$

d. 0_k

f. $\begin{bmatrix} X^m & 0 \\ 0 & X^m \end{bmatrix}$ if $n = 2m$; $\begin{bmatrix} 0 & X^{m+1} \\ X^m & 0 \end{bmatrix}$ if $n = 2m+1$

2.3.14 b. If Y is row i of the identity matrix I, then YA is row i of $IA = A$.

2.3.16 b. $AB - BA$

d. 0

2.3.18 b. $(kA)C = k(AC) = k(CA) = C(kA)$

2.3.20 We have $A^T = A$ and $B^T = B$, so $(AB)^T = B^TA^T = BA$. Hence AB is symmetric if and only if $AB = BA$.

2.3.22 b. $A = 0$

2.3.24 If $BC = I$, then $AB = 0$ gives $0 = 0C = (AB)C = A(BC) = AI = A$, contrary to the assumption that $A \neq 0$.

2.3.26 3 paths $v_1 \to v_4$, 0 paths $v_2 \to v_3$

2.3.27 b. False. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = J$, then $AJ = A$ but $J \neq I$.

d. True. Since $A^T = A$, we have $(I + AT = I^T + A^T = I + A$.

f. False. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A \neq 0$ but $A^2 = 0$.

h. True. We have $A(A+B) = (A+B)A$; that is, $A^2 + AB = A^2 + BA$. Subtracting A^2 gives $AB = BA$.

j. False. $A = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

l. False. See **(j)**.

2.3.28 b. If $A = [a_{ij}]$ and $B = [b_{ij}]$ and $\sum_j a_{ij} = 1 = \sum_j b_{ij}$, then the $(i,\ j)$-entry of AB is $c_{ij} = \sum_k a_{ik} b_{kj}$, whence $\sum_j c_{ij} = \sum_j \sum_k a_{ik} b_{kj} = \sum_k a_{ik} (\sum_j b_{kj}) = \sum_k a_{ik} = 1$. Alternatively: If $\mathbf{e} = (1,\ 1,\ \dots,\ 1)$, then the rows of A sum to 1 if and only if $A\mathbf{e} = \mathbf{e}$. If also $B\mathbf{e} = \mathbf{e}$ then $(AB)\mathbf{e} = A(B\mathbf{e}) = A\mathbf{e} = \mathbf{e}$.

2.3.30 b. If $A = [a_{ij}]$, then $\text{tr}\,(kA) = \text{tr}\ [ka_{ij}] = \sum_{i=1}^{n} ka_{ii} = k \sum_{i=1}^{n} a_{ii} = k\ \text{tr}\,(A)$.

e. Write $A^T = \left[a'_{ij}\right]$, where $a'_{ij} = a_{ji}$. Then $AA^T = \left(\sum_{k=1}^{n} a_{ik} a'_{kj}\right)$, so $\text{tr}\,(AA^T) = \sum_{i=1}^{n} \left[\sum_{k=1}^{n} a_{ik} a'_{ki}\right] = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2$.

2.3.32 e. Observe that $PQ = P^2 + PAP - P^2AP = P$, so $Q^2 = PQ + APQ - PAPQ = P + AP - PAP = Q$.

2.3.34 b. $(A+B)(A-B) = A^2 - AB + BA - B^2$, and $(A-B)(A+B) = A^2 + AB - BA - B^2$. These are equal if and only if $-AB + BA = AB - BA$; that is, $2BA = 2AB$; that is, $BA = AB$.

2.3.35 b. $(A+B)(A-B) = A^2 - AB + BA - B^2$ and $(A-B)(A+B) = A^2 - BA + AB - B^2$. These are equal if and only if $-AB + BA = -BA + AB$, that is $2AB = 2BA$, that is $AB = BA$.

Section 2.4

2.4.2 b. $\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

f. $\frac{1}{10}\begin{bmatrix} 1 & 4 & -1 \\ -2 & 2 & 2 \\ -9 & 14 & -1 \end{bmatrix}$

h. $\frac{1}{4}\begin{bmatrix} 2 & 0 & -2 \\ -5 & 2 & 5 \\ -3 & 2 & -1 \end{bmatrix}$

j. $\begin{bmatrix} 0 & 0 & 1 & -2 \\ -1 & -2 & -1 & -3 \\ 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

l. $\begin{bmatrix} 1 & -2 & 6 & -30 & 210 \\ 0 & 1 & -3 & 15 & -105 \\ 0 & 0 & 1 & -5 & 35 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2.4.3 b. $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 \\ -2 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 9 & -14 & 6 \\ 4 & -4 & 1 \\ -10 & 15 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 23 \\ 8 \\ -25 \end{bmatrix}$

2.4.4 b. $B = A^{-1}AB = \begin{bmatrix} 4 & -2 & 1 \\ 7 & -2 & 4 \\ -1 & 2 & -1 \end{bmatrix}$

2.4.5 b. $\frac{1}{10}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

d. $\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

f. $\frac{1}{2}\begin{bmatrix} 2 & 0 \\ -6 & 1 \end{bmatrix}$

h. $-\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

2.4.6 b. $A = \frac{1}{2}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & -1 \end{bmatrix}$

2.4.8 b. A and B are inverses.

2.4.9 b. False. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

d. True. $A^{-1} = \frac{1}{3}A^3$

f. False. $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

h. True. If $(A^2)B = I$, then $A(AB) = I$; use Theorem 2.4.5.

2.4.10 b. $(C^T)^{-1} = (C^{-1})^T = A^T$ because $C^{-1} = (A^{-1})^{-1} = A$.

2.4.11 b. (i) Inconsistent.

(ii) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

2.4.15 b. $B^4 = I$, so $B^{-1} = B^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

2.4.16 $\begin{bmatrix} c^2-2 & -c & 1 \\ -c & 1 & 0 \\ 3-c^2 & c & -1 \end{bmatrix}$

2.4.18 b. If column j of A is zero, $A\mathbf{y} = \mathbf{0}$ where $\mathbf{y}$ is column j of the identity matrix. Use Theorem 2.4.5.

d. If each column of A sums to 0, $XA = 0$ where X is the row of 1s. Hence $A^T X^T = 0$ so A has no inverse by Theorem 2.4.5 ($X^T \neq 0$).

2.4.19 b. (ii) $(-1,\ 1,\ 1)A = 0$

2.4.20 b. Each power A^k is invertible by Theorem 2.4.4 (because A is invertible). Hence A^k cannot be 0.

2.4.21 b. By (a), if one has an inverse the other is zero and so has no inverse.

2.4.22 If $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, $a > 1$, then $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}$ is an x-compression because $\frac{1}{a} < 1$.

2.4.24 b. $A^{-1} = \frac{1}{4}(A^3 + 2A^2 - 1)$

2.4.25 b. If $B\mathbf{x} = \mathbf{0}$, then $(AB)\mathbf{x} = (A)B\mathbf{x} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$ because AB is invertible. Hence B is invertible by Theorem 2.4.5. But then $A = (AB)B^{-1}$ is invertible by Theorem 2.4.4.

2.4.26 b. $\left[\begin{array}{rr|r} 2 & -1 & 0 \\ -5 & 3 & 0 \\ \hline -13 & 8 & -1 \end{array}\right]$

d. $\left[\begin{array}{rr|rr} 1 & -1 & -14 & 8 \\ -1 & 2 & 16 & -9 \\ \hline 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

2.4.28 d. If $A^n = 0$, $(I - A)^{-1} = I + A + \cdots + A^{n-1}$.

2.4.30 b. $A[B(AB)^{-1}] = I = [(BA)^{-1}B]A$, so A is invertible by Exercise 2.4.10.

2.4.32 a. Have $AC = CA$. Left-multiply by A^{-1} to get $C = A^{-1}CA$. Then right-multiply by A^{-1} to get $CA^{-1} = A^{-1}C$.

2.4.33 b. Given $ABAB = AABB$. Left multiply by A^{-1}, then right multiply by B^{-1}.

2.4.34 If $B\mathbf{x} = \mathbf{0}$ where $\mathbf{x}$ is $n \times 1$, then $AB\mathbf{x} = \mathbf{0}$ so $\mathbf{x} = \mathbf{0}$ as AB is invertible. Hence B is invertible by Theorem 2.4.5, so $A = (AB)B^{-1}$ is invertible.

2.4.35 b. $B\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = 0$ so B is not invertible by Theorem 2.4.5.

2.4.38 b. Write $U = I_n - 2XX^T$. Then $U^T = I_n^T - 2X^{TT}X^T = U$, and $U^2 = I_n^2 - (2XX^T)I_n - I_n(2XX^T) + 4(XX^T)(XX^T) = I_n - 4XX^T + 4XX^T = I_n$.

2.4.39 b. $(I - 2P)^2 = I - 4P + 4P^2$, and this equals I if and only if $P^2 = P$.

2.4.41 b. $(A^{-1} + B^{-1})^{-1} = B(A+B)^{-1}A$

Section 2.5

2.5.1 b. Interchange rows 1 and 3 of I. $E^{-1} = E$.

d. Add (-2) times row 1 of I to row 2.
$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

f. Multiply row 3 of I by 5. $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$

2.5.2 b. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2.5.3 b. The only possibilities for E are $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$. In each case, EA has a row different from C.

2.5.5 b. No, 0 is not invertible.

2.5.6 b. $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$
$A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \end{bmatrix}$. Alternatively,
$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$
$A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \end{bmatrix}$.

d. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{7}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2.5.7 b. $U = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2.5.8 b. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

2.5.10 $UA = R$ by Theorem 2.5.1, so $A = U^{-1}R$.

2.5.12 b. $U = A^{-1}$, $V = I^2$; rank $A = 2$

d. $U = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$,
$V = \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; rank $A = 2$

2.5.16 Write $U^{-1} = E_kE_{k-1}\cdots E_2E_1$, E_i elementary. Then $\begin{bmatrix} I & U^{-1}A \end{bmatrix} = \begin{bmatrix} U^{-1}U & U^{-1}A \end{bmatrix}$ $= U^{-1}\begin{bmatrix} U & A \end{bmatrix} = E_kE_{k-1}\cdots E_2E_1\begin{bmatrix} U & A \end{bmatrix}$. So $\begin{bmatrix} U & A \end{bmatrix} \to \begin{bmatrix} I & U^{-1}A \end{bmatrix}$ by row operations (Lemma 2.5.1).

2.5.17 b. (i) $A \overset{r}{\sim} A$ because $A = IA$. (ii) If $A \overset{r}{\sim} B$, then $A = UB$, U invertible, so $B = U^{-1}A$. Thus $B \overset{r}{\sim} A$. (iii) If $A \overset{r}{\sim} B$ and $B \overset{r}{\sim} C$, then $A = UB$ and $B = VC$, U and V invertible. Hence $A = U(VC) = (UV)C$, so $A \overset{r}{\sim} C$.

2.5.19 b. If $B \overset{r}{\sim} A$, let $B = UA$, U invertible. If $U = \begin{bmatrix} d & b \\ -b & d \end{bmatrix}$, $B = UA = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & d \end{bmatrix}$ where b and d are not both zero (as U is invertible). Every such matrix B arises in this way: Use $U = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$–it is invertible by Example 2.3.5.

2.5.22 b. Multiply column i by $1/k$.

Section 2.6

2.6.1 b. $\begin{bmatrix} 5 \\ 6 \\ -13 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$, so
$T\begin{bmatrix} 5 \\ 6 \\ -13 \end{bmatrix} = 3T\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - 2T\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} =$
$3\begin{bmatrix} 3 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \end{bmatrix}$

2.6.2 b. As in 1(b), $T\begin{bmatrix} 5 \\ -1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -9 \end{bmatrix}$.

2.6.3 b. $T(\mathbf{e}_1) = -\mathbf{e}_2$ and $T(\mathbf{e}_2) = -\mathbf{e}_1$. So $A\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} -\mathbf{e}_2 & -\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

d. $T(\mathbf{e}_1) = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

So $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

2.6.4 b. $T(\mathbf{e}_1) = -\mathbf{e}_1$, $T(\mathbf{e}_2) = \mathbf{e}_2$ and $T(\mathbf{e}_3) = \mathbf{e}_3$. Hence Theorem 2.6.2 gives $A\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.6.5 b. We have $\mathbf{y}_1 = T(\mathbf{x}_1)$ for some $\mathbf{x}_1$ in $\mathbb{R}^n$, and $\mathbf{y}_2 = T(\mathbf{x}_2)$ for some $\mathbf{x}_2$ in $\mathbb{R}^n$. So $a\mathbf{y}_1 + b\mathbf{y}_2 = aT(\mathbf{x}_1) + bT(\mathbf{x}_2) = T(a\mathbf{x}_1 + b\mathbf{x}_2)$. Hence $a\mathbf{y}_1 + b\mathbf{y}_2$ is also in the image of T.

2.6.7 b. $T\left(2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq 2\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

2.6.8 b. $A = \frac{1}{\sqrt{2}} \left[\begin{array}{rr} 1 & 1 \\ -1 & 1 \end{array} \right]$, rotation through $\theta = -\frac{\pi}{4}$.

d. $A = \frac{1}{10} \left[\begin{array}{rr} -8 & -6 \\ -6 & 8 \end{array} \right]$, reflection in the line $y = -3x$.

2.6.10 b. $\left[\begin{array}{ccc} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{array} \right]$

2.6.12 b. Reflection in the y axis

d. Reflection in $y = x$

f. Rotation through $\frac{\pi}{2}$

2.6.13 b. $T(\mathbf{x}) = aR(\mathbf{x}) = a(A\mathbf{x}) = (aA)\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}$. Hence T is induced by aA.

2.6.14 b. If $\mathbf{x}$ is in $\mathbb{R}^n$, then $T(-\mathbf{x}) = T[(-1)\mathbf{x}] = (-1)T(\mathbf{x}) = -T(\mathbf{x})$.

2.6.17 b. If $B^2 = I$ then $T^2(\mathbf{x}) = T[T(\mathbf{x})] = B(B\mathbf{x}) = B^2\mathbf{x} = I\mathbf{x} = \mathbf{x} = 1_{\mathbb{R}^2}(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Hence $T^2 = 1_{\mathbb{R}^2}$. If $T^2 = 1_{\mathbb{R}^2}$, then $B^2\mathbf{x} = T^2(\mathbf{x}) = 1_{\mathbb{R}^2}(\mathbf{x}) = \mathbf{x} = I\mathbf{x}$ for all $\mathbf{x}$, so $B^2 = I$ by Theorem 2.2.6.

2.6.18 b. The matrix of $Q_1 \circ Q_0$ is $\left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right] = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array} \right]$, which is the matrix of $R_{\frac{\pi}{2}}$.

d. The matrix of $Q_0 \circ R_{\frac{\pi}{2}}$ is $\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right] \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array} \right] = \left[\begin{array}{rr} 0 & -1 \\ -1 & 0 \end{array} \right]$, which is the matrix of Q_{-1}.

2.6.20 We have

$$T(\mathbf{x}) = x_1 + x_2 + \cdots + x_n = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right]$$, so T

is the matrix transformation induced by the matrix $A = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array} \right]$. In particular, T is linear. On the other hand, we can use Theorem 2.6.2 to get A, but to do this we must first show directly that T is linear. If we write $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right]$ and $\mathbf{y} = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \right]$. Then

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= T \left[\begin{array}{c} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{array} \right] \\ &= (x_1+y_1) + (x_2+y_2) + \cdots + (x_n+y_n) \\ &= (x_1+x_2+\cdots+x_n) + (y_1+y_2+\cdots+y_n) \\ &= T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

Similarly, $T(a\mathbf{x}) = aT(\mathbf{x})$ for any scalar a, so T is linear. By Theorem 2.6.2, T has matrix $A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array} \right] = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array} \right]$, as before.

2.6.22 b. If $T : \mathbb{R}^n \to \mathbb{R}$ is linear, write $T(\mathbf{e}_j) = w_j$ for each $j = 1, 2, \dots, n$ where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. Since $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$, Theorem 2.6.1 gives

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n) \\ &= x_1w_1 + x_2w_2 + \cdots + x_nw_n \\ &= \mathbf{w} \cdot \mathbf{x} = T_{\mathbf{w}}(\mathbf{x}) \end{aligned}$$

where $\mathbf{w} = \left[\begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_n \end{array} \right]$. Since this holds for all $\mathbf{x}$ in $\mathbb{R}^n$, it shows that $T = T_{\mathbf{W}}$. This also follows from Theorem 2.6.2, but we have first to verify that T is linear. (This comes to showing that $\mathbf{w} \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{w} \cdot \mathbf{s} + \mathbf{w} \cdot \mathbf{y}$ and $\mathbf{w} \cdot (a\mathbf{x}) = a(\mathbf{w} \cdot \mathbf{x})$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all a in $\mathbb{R}$.) Then T has matrix $A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array} \right] = \left[\begin{array}{cccc} w_1 & w_2 & \cdots & w_n \end{array} \right]$ by Theorem 2.6.2. Hence if $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right]$ in $\mathbb{R}$, then $T(\mathbf{x}) = A\mathbf{x} = \mathbf{w} \cdot \mathbf{x}$, as required.

2.6.23 b. Given $\mathbf{x}$ in $\mathbb{R}$ and a in $\mathbb{R}$, we have

$(S \circ T)(a\mathbf{x})$	$=$	$S[T(a\mathbf{x})]$	Definition of $S \circ T$
	$=$	$S[aT(\mathbf{x})]$	Because T is linear.
	$=$	$a[S[T(\mathbf{x})]]$	Because S is linear.
	$=$	$a[S \circ T(\mathbf{x})]$	Definition of $S \circ T$

Section 2.7

2.7.1 b. $\left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & -3 & 0 \\ -1 & 9 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{array} \right]$

d. $\left[\begin{array}{rrrr} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{array} \right] \left[\begin{array}{rrrrr} 1 & 3 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$

f. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2.7.2 b. $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$PA = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

d. $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$PA = \begin{bmatrix} -1 & -2 & 3 & 0 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \\ 2 & 4 & -6 & 5 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & -2 & 0 \\ 2 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.7.3 b. $\mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} -1+2t \\ -t \\ s \\ t \end{bmatrix}$ s and t arbitrary

d. $\mathbf{y} = \begin{bmatrix} 2 \\ 8 \\ -1 \\ 0 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} 8-2t \\ 6-t \\ -1-t \\ t \end{bmatrix}$ t arbitrary

2.7.5 $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \to \begin{bmatrix} R_1+R_2 \\ R_2 \end{bmatrix} \to \begin{bmatrix} R_1+R_2 \\ -R_1 \end{bmatrix} \to \begin{bmatrix} R_2 \\ -R_1 \end{bmatrix} \to \begin{bmatrix} R_2 \\ R_1 \end{bmatrix}$

2.7.6 b. Let $A = LU = L_1U_1$ be LU-factorizations of the invertible matrix A. Then U and U_1 have no row of zeros and so (being row-echelon) are upper triangular with 1's on the main diagonal. Thus, using **(a.)**, the diagonal matrix $D = UU_1^{-1}$ has 1's on the main diagonal. Thus $D = I$, $U = U_1$, and $L = L_1$.

2.7.7 If $A = \begin{bmatrix} a & 0 \\ X & A_1 \end{bmatrix}$ and $B = \begin{bmatrix} b & 0 \\ Y & B_1 \end{bmatrix}$ in block form, then $AB = \begin{bmatrix} ab & 0 \\ Xb+A_1Y & A_1B_1 \end{bmatrix}$, and A_1B_1 is lower triangular by induction.

2.7.9 b. Let $A = LU = L_1U_1$ be two such factorizations. Then $UU_1^{-1} = L^{-1}L_1$; write this matrix as $D = UU_1^{-1} = L^{-1}L_1$. Then D is lower triangular (apply Lemma 2.7.1 to $D = L^{-1}L_1$); and D is also upper triangular (consider UU_1^{-1}). Hence D is diagonal, and so $D = I$ because L^{-1} and L_1 are unit triangular. Since $A = LU$; this completes the proof.

Section 2.8

2.8.1 b. $\begin{bmatrix} t \\ 3t \\ t \end{bmatrix}$

d. $\begin{bmatrix} 14t \\ 17t \\ 47t \\ 23t \end{bmatrix}$

2.8.2 $\begin{bmatrix} t \\ t \\ t \end{bmatrix}$

2.8.4 $P = \begin{bmatrix} bt \\ (1-a)t \end{bmatrix}$ is nonzero (for some t) unless $b = 0$ and $a = 1$. In that case, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution. If the entries of E are positive, then $P = \begin{bmatrix} b \\ 1-a \end{bmatrix}$ has positive entries.

2.8.7 b. $\begin{bmatrix} 0.4 & 0.8 \\ 0.7 & 0.2 \end{bmatrix}$

2.8.8 If $E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I - E = \begin{bmatrix} 1-a & -b \\ -c & 1-d \end{bmatrix}$, so $\det(I-E) = (1-a)(1-d) - bc = 1 - \operatorname{tr} E + \det E$. If $\det(I-E) \neq 0$, then $(I-E)^{-1} = \frac{1}{\det(I-E)} \begin{bmatrix} 1-d & b \\ c & 1-a \end{bmatrix}$, so $(I-E)^{-1} \geq 0$ if $\det(I-E) > 0$, that is, $\operatorname{tr} E < 1 + \det E$. The converse is now clear.

2.8.9 b. Use $\mathbf{p} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ in Theorem 2.8.2.

d. $\mathbf{p} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ in Theorem 2.8.2.

Section 2.9

2.9.1 b. Not regular

2.9.2 b. $\frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\frac{3}{8}$

d. $\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, 0.312

f. $\frac{1}{20}\begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$, 0.306

2.9.4 b. 50% middle, 25% upper, 25% lower

2.9.6 $\frac{7}{16}$, $\frac{9}{16}$

2.9.8 a. $\frac{7}{75}$

b. He spends most of his time in compartment 3; steady state $\frac{1}{16}\begin{bmatrix} 3 \\ 2 \\ 5 \\ 4 \\ 2 \end{bmatrix}$.

2.9.12 a. Direct verification.

b. Since $0 < p < 1$ and $0 < q < 1$ we get $0 < p + q < 2$ whence $-1 < p + q - 1 < 1$. Finally, $-1 < 1 - p - q < 1$, so $(1 - p - q)^m$ converges to zero as m increases.

Supplementary Exercises for Chapter 2

Supplementary Exercise 2.2. b. $U^{-1} = \frac{1}{4}(U^2 - 5U + 11I)$.

Supplementary Exercise 2.4. b. If $\mathbf{x}_k = \mathbf{x}_m$, then $\mathbf{y} + k(\mathbf{y} - \mathbf{z}) = \mathbf{y} + m(\mathbf{y} - \mathbf{z})$. So $(k - m)(\mathbf{y} - \mathbf{z}) = \mathbf{0}$. But $\mathbf{y} - \mathbf{z}$ is not zero (because $\mathbf{y}$ and $\mathbf{z}$ are distinct), so $k - m = 0$ by Example 2.1.7.

Supplementary Exercise 2.6. d. Using parts **(c)** and **(b)** gives $I_{pq}AI_{rs} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}I_{pq}I_{ij}I_{rs}$. The only nonzero term occurs when $i = q$ and $j = r$, so $I_{pq}AI_{rs} = a_{qr}I_{ps}$.

Supplementary Exercise 2.7. b. If $A = [a_{ij}] = \sum_{ij} a_{ij}I_{ij}$, then $I_{pq}AI_{rs} = a_{qr}I_{ps}$ by 6(d). But then $a_{qr}I_{ps} = AI_{pq}I_{rs} = 0$ if $q \neq r$, so $a_{qr} = 0$ if $q \neq r$. If $q = r$, then $a_{qq}I_{ps} = AI_{pq}I_{rs} = AI_{ps}$ is independent of q. Thus $a_{qq} = a_{11}$ for all q.

Section 3.1

3.1.1 b. 0

d. -1

f. -39

h. 0

j. $2abc$

l. 0

n. -56

p. $abcd$

3.1.5 b. -17

d. 106

3.1.6 b. 0

3.1.7 b. 12

3.1.8 b.
$$\det\begin{bmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{bmatrix}$$
$$= 3\det\begin{bmatrix} a+p+x & b+q+y & c+r+z \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{bmatrix}$$
$$= 3\det\begin{bmatrix} a+p+x & b+q+y & c+r+z \\ p-a & q-b & r-c \\ x-p & y-q & z-r \end{bmatrix}$$
$$= 3\det\begin{bmatrix} 3x & 3y & 3z \\ p-a & q-b & r-c \\ x-p & y-q & z-r \end{bmatrix}\cdots$$

3.1.9 b. False. $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

d. False. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \to R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

f. False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

h. False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

3.1.10 b. 35

3.1.11 b. -6

d. -6

3.1.14 b. $-(x-2)(x^2+2x-12)$

3.1.15 b. -7

3.1.16 b. $\pm\frac{\sqrt{6}}{2}$

d. $x = \pm y$

3.1.21 Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{x}+\mathbf{y} & \cdots & \mathbf{c}_n \end{bmatrix}$ where $\mathbf{x}+\mathbf{y}$ is in column j. Expanding $\det A$ along column j (the one containing $\mathbf{x}+\mathbf{y}$):

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) = \det A &= \sum_{i=1}^{n}(x_i+y_i)c_{ij}(A) \\ &= \sum_{i=1}^{n} x_i c_{ij}(A) + \sum_{i=1}^{n} y_i c_{ij}(A) \\ &= T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

Similarly for $T(a\mathbf{x}) = aT(\mathbf{x})$.

3.1.24 If A is $n \times n$, then $\det B = (-1)^k \det A$ where $n = 2k$ or $n = 2k+1$.

Section 3.2

3.2.1 b. $\begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 6 \\ -3 & 1 & 4 \end{bmatrix}$

d. $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = A$

3.2.2 b. $c \neq 0$

d. any c

f. $c \neq -1$

3.2.3 b. -2

3.2.4 b. 1

3.2.6 b. $\frac{4}{9}$

3.2.7 b. 16

3.2.8 b. $\frac{1}{11}\begin{bmatrix} 5 \\ 21 \end{bmatrix}$

d. $\frac{1}{79}\begin{bmatrix} 12 \\ -37 \\ -2 \end{bmatrix}$

3.2.9 b. $\frac{4}{51}$

3.2.10 b. $\det A = 1,\ -1$

d. $\det A = 1$

f. $\det A = 0$ if n is odd; nothing can be said if n is even

3.2.15 dA where $d = \det A$

3.2.19 b. $\frac{1}{c}\begin{bmatrix} 1 & 0 & 1 \\ 0 & c & 1 \\ -1 & c & 1 \end{bmatrix}$, $c \neq 0$

d. $\frac{1}{2}\begin{bmatrix} 8-c^2 & -c & c^2-6 \\ c & 1 & -c \\ c^2-10 & c & 8-c^2 \end{bmatrix}$

f. $\frac{1}{c^3+1}\begin{bmatrix} 1-c & c^2+1 & -c-1 \\ c^2 & -c & c+1 \\ -c & 1 & c^2-1 \end{bmatrix}$, $c \neq -1$

3.2.20 b. T. $\det AB = \det A \det B = \det B \det A = \det BA$.

d. T. $\det A \neq 0$ means A^{-1} exists, so $AB = AC$ implies that $B = C$.

f. F. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ then $\text{adj } A = 0$.

h. F. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ then $\text{adj } A = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$

j. F. If $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ then $\det(I+A) = -1$ but $1 + \det A = 1$.

l. F. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then $\det A = 1$ but $\text{adj } A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \neq A$

3.2.22 b. $5 - 4x + 2x^2$.

3.2.23 b. $1 - \frac{5}{3}x + \frac{1}{2}x^2 + \frac{7}{6}x^3$

3.2.24 b. $1 - 0.51x + 2.1x^2 - 1.1x^3$; 1.25, so $y = 1.25$

3.2.26 b. Use induction on n where A is $n \times n$. It is clear if $n = 1$. If $n > 1$, write $A = \begin{bmatrix} a & X \\ 0 & B \end{bmatrix}$ in block form where B is $(n-1) \times (n-1)$. Then $A^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$, and this is upper triangular because B is upper triangular by induction.

3.2.28 $-\frac{1}{21}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$

3.2.34 b. Have $(\text{adj } A)A = (\det A)I$; so taking inverses, $A^{-1} \cdot (\text{adj } A)^{-1} = \frac{1}{\det A}I$. On the other hand, $A^{-1} \text{ adj } (A^{-1}) = \det(A^{-1})I = \frac{1}{\det A}I$. Comparison yields $A^{-1}(\text{adj } A)^{-1} = A^{-1} \text{ adj } (A^{-1})$, and part **(b)** follows.

d. Write $\det A = d$, $\det B = e$. By the adjugate formula $AB \text{ adj } (AB) = deI$, and $AB \text{ adj } B \text{ adj } A = A[eI] \text{ adj } A = (eI)(dI) = deI$. Done as AB is invertible.

Section 3.3

3.3.1 b. $(x-3)(x+2); 3; -2; \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $P = \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$; $P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

d. $(x-2)^3; 2; \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$; No such P; Not diagonalizable.

f. $(x+1)^2(x-2); -1, -2; \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$; No such P; Not diagonalizable. Note that this matrix and the matrix in Example 3.3.9 have the same characteristic polynomial, but that matrix is diagonalizable.

h. $(x-1)^2(x-3); 1, 3; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ No such P; Not diagonalizable.

3.3.2 b. $V_k = \frac{7}{3}2^k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

d. $V_k = \frac{3}{2}3^k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3.3.4 $A\mathbf{x} = \lambda\mathbf{x}$ if and only if $(A - \alpha I)\mathbf{x} = (\lambda - \alpha)\mathbf{x}$. Same eigenvectors.

3.3.8 b. $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so
$A^n = P \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} P^{-1} = \begin{bmatrix} 9 - 8 \cdot 2^n & 12(1-2^n) \\ 6(2^n - 1) & 9 \cdot 2^n - 8 \end{bmatrix}$

3.3.9 b. $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

3.3.11 b. and d. $PAP^{-1} = D$ is diagonal, then b. $P^{-1}(kA)P = kD$ is diagonal, and d. $Q(U^{-1}AU)Q = D$ where $Q = PU$.

3.3.12 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable by Example 3.3.8. But $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ where $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ has diagonalizing matrix $P = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ is already diagonal.

3.3.14 We have $\lambda^2 = \lambda$ for every eigenvalue λ (as $\lambda = 0, 1$) so $D^2 = D$, and so $A^2 = A$ as in Example 3.3.9.

3.3.18 b. $c_{rA}(x) = \det[xI - rA]$
$= r^n \det\left[\frac{x}{r}I - A\right] = r^n c_A\left[\frac{x}{r}\right]$

3.3.20 b. If $\lambda \neq 0$, $A\mathbf{x} = \lambda\mathbf{x}$ if and only if $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. The result follows.

3.3.21 b. $(A^3 - 2A - 3I)\mathbf{x} = A^3\mathbf{x} - 2A\mathbf{x} + 3\mathbf{x} = \lambda^3\mathbf{x} - 2\lambda\mathbf{x} + 3\mathbf{x} = (\lambda^3 - 2\lambda - 3)\mathbf{x}$.

3.3.23 b. If $A^m = 0$ and $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$, then $A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. In general, $A^k\mathbf{x} = \lambda^k\mathbf{x}$ for all $k \geq 1$. Hence, $\lambda^m\mathbf{x} = A^m\mathbf{x} = \mathbf{0}\mathbf{x} = \mathbf{0}$, so $\lambda = 0$ (because $\mathbf{x} \neq \mathbf{0}$).

3.3.24 a. If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^k\mathbf{x} = \lambda^k\mathbf{x}$ for each k. Hence $\lambda^m\mathbf{x} = A^m\mathbf{x} = \mathbf{x}$, so $\lambda^m = 1$. As λ is real, $\lambda = \pm 1$ by the Hint. So if $P^{-1}AP = D$ is diagonal, then $D^2 = I$ by Theorem 3.3.4. Hence $A^2 = PD^2P = I$.

3.3.27 a. We have $P^{-1}AP = \lambda I$ by the diagonalization algorithm, so $A = P(\lambda I)P^{-1} = \lambda PP^{-1} = \lambda I$.

b. No. $\lambda = 1$ is the only eigenvalue.

3.3.31 b. $\lambda_1 = 1$, stabilizes.

d. $\lambda_1 = \frac{1}{24}(3 + \sqrt{69}) = 1.13$, diverges.

3.3.34 Extinct if $\alpha < \frac{1}{5}$, stable if $\alpha = \frac{1}{5}$, diverges if $\alpha > \frac{1}{5}$.

Section 3.4

3.4.1 b. $x_k = \frac{1}{3}\left[4 - (-2)^k\right]$

d. $x_k = \frac{1}{5}\left[2^{k+2} + (-3)^k\right]$

3.4.2 b. $x_k = \frac{1}{2}\left[(-1)^k + 1\right]$

3.4.3 b. $x_{k+4} = x_k + x_{k+2} + x_{k+3}; x_{10} = 169$

3.4.5 $\frac{1}{2\sqrt{5}}\left[3 + \sqrt{5}\right]\lambda_1^k + (-3 + \sqrt{5})\lambda_2^k$ where $\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 - \sqrt{5})$.

3.4.7 $\frac{1}{2\sqrt{3}}\left[2+\sqrt{3}\right]\lambda_1^k+(-2+\sqrt{3})\lambda_2^k$ where $\lambda_1=1+\sqrt{3}$ and $\lambda_2=1-\sqrt{3}$.

3.4.9 $\frac{34}{3}-\frac{4}{3}\left(-\frac{1}{2}\right)^k$. Long term $11\frac{1}{3}$ million tons.

3.4.11 b. $A\left[\begin{array}{c}1\\ \lambda\\ \lambda^2\end{array}\right]=\left[\begin{array}{c}\lambda\\ \lambda^2\\ a+b\lambda+c\lambda^2\end{array}\right]=\left[\begin{array}{c}\lambda\\ \lambda^2\\ \lambda^3\end{array}\right]=\lambda\left[\begin{array}{c}1\\ \lambda\\ \lambda^2\end{array}\right]$

3.4.12 b. $x_k=\frac{11}{10}3^k+\frac{11}{15}(-2)^k-\frac{5}{6}$

3.4.13 a. $p_{k+2}+q_{k+2}=[ap_{k+1}+bp_k+c(k)]+[aq_{k+1}+bq_k]=a(p_{k+1}+q_{k+1})+b(p_k+q_k)+c(k)$

Section 3.5

3.5.1 b. $c_1\left[\begin{array}{c}1\\1\end{array}\right]e^{4x}+c_2\left[\begin{array}{c}5\\-1\end{array}\right]e^{-2x}; c_1=-\frac{2}{3},\ c_2=\frac{1}{3}$

d. $c_1\left[\begin{array}{c}-8\\10\\7\end{array}\right]e^{-x}+c_2\left[\begin{array}{c}1\\-2\\1\end{array}\right]e^{2x}+c_3\left[\begin{array}{c}1\\0\\1\end{array}\right]e^{4x}$; $c_1=0,\ c_2=-\frac{1}{2},\ c_3=\frac{3}{2}$

3.5.3 b. The solution to (a) is $m(t)=10\left(\frac{4}{5}\right)^{t/3}$. Hence we want t such that $10\left(\frac{4}{5}\right)^{t/3}=5$. We solve for t by taking natural logarithms:

$$t=\frac{3\ln(\frac{1}{2})}{\ln(\frac{4}{5})}=9.32 \text{ hours.}$$

3.5.5 a. If $\mathbf{g}'=A\mathbf{g}$, put $\mathbf{f}=\mathbf{g}-A^{-1}\mathbf{b}$. Then $\mathbf{f}'=\mathbf{g}'$ and $A\mathbf{f}=A\mathbf{g}-\mathbf{b}$, so $\mathbf{f}'=\mathbf{g}'=A\mathbf{g}=A\mathbf{f}+\mathbf{b}$, as required.

3.5.6 b. Assume that $f_1'=a_1f_1+f_2$ and $f_2'=a_2f_1$. Differentiating gives $f_1''=a_1f_1'+f_2'=a_1f_1'+a_2f_1$, proving that f_1 satisfies Equation 3.15.

Section 3.6

3.6.2 Consider the rows $R_p, R_{p+1}, \dots, R_{q-1}, R_q$. In $q-p$ adjacent interchanges they can be put in the order $R_{p+1}, \dots, R_{q-1}, R_q, R_p$. Then in $q-p-1$ adjacent interchanges we can obtain the order $R_q, R_{p+1}, \dots, R_{q-1}, R_p$. This uses $2(q-p)-1$ adjacent interchanges in all.

Supplementary Exercises for Chapter 3

Supplementary Exercise 3.2. b. If A is 1×1, then $A^T=A$. In general, $\det\left[A_{ij}\right]=\det\left[(A_{ij})^T\right]=\det\left[(A^T)_{ji}\right]$ by (a) and induction. Write $A^T=\left[a'_{ij}\right]$ where $a'_{ij}=a_{ji}$, and expand $\det A^T$ along column 1.

$$\begin{aligned}\det A^T &= \sum_{j=1}^{n} a'_{j1}(-1)^{j+1}\det\left[(A^T)_{j1}\right]\\ &= \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\det\left[A_{1j}\right]=\det A\end{aligned}$$

where the last equality is the expansion of $\det A$ along row 1.

Section 4.1

4.1.1 b. $\sqrt{6}$

d. $\sqrt{5}$

f. $3\sqrt{6}$

4.1.2 b. $\frac{1}{3}\left[\begin{array}{c}-2\\-1\\2\end{array}\right]$

4.1.4 b. $\sqrt{2}$

d. 3

4.1.6 b. $\overrightarrow{FE}=\overrightarrow{FC}+\overrightarrow{CE}=\frac{1}{2}\overrightarrow{AC}+\frac{1}{2}\overrightarrow{CB}=\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{CB})=\frac{1}{2}\overrightarrow{AB}$

4.1.7 b. Yes

d. Yes

4.1.8 b. $\mathbf{p}$

d. $-(\mathbf{p}+\mathbf{q})$.

4.1.9 b. $\left[\begin{array}{c}-1\\-1\\5\end{array}\right]$, $\sqrt{27}$

d. $\left[\begin{array}{c}0\\0\\0\end{array}\right]$, 0

f. $\left[\begin{array}{c}-2\\2\\2\end{array}\right]$, $\sqrt{12}$

4.1.10 b. **(i)** $Q(5,\ -1,\ 2)$ **(ii)** $Q(1,\ 1,\ -4)$.

4.1.11 b. $\mathbf{x}=\mathbf{u}-6\mathbf{v}+5\mathbf{w}=\left[\begin{array}{c}-26\\4\\19\end{array}\right]$

4.1.12 b. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 6 \end{bmatrix}$

4.1.13 b. If it holds then $\begin{bmatrix} 3a+4b+c \\ -a+c \\ b+c \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\left[\begin{array}{rrr|r} 3 & 4 & 1 & x_1 \\ -1 & 0 & 1 & x_2 \\ 0 & 1 & 1 & x_3 \end{array}\right] \to \left[\begin{array}{rrr|r} 0 & 4 & 4 & x_1+3x_2 \\ -1 & 0 & 1 & x_2 \\ 0 & 1 & 1 & x_3 \end{array}\right]$$

If there is to be a solution then $x_1+3x_2 = 4x_3$ must hold. This is not satisfied.

4.1.14 b. $\frac{1}{4}\begin{bmatrix} 5 \\ -5 \\ -2 \end{bmatrix}$

4.1.17 b. $Q(0, 7, 3)$.

4.1.18 b. $\mathbf{x} = \frac{1}{40}\begin{bmatrix} -20 \\ -13 \\ 14 \end{bmatrix}$

4.1.20 b. $S(-1, 3, 2)$.

4.1.21 b. T. $\|\mathbf{v}-\mathbf{w}\| = 0$ implies that $\mathbf{v}-\mathbf{w} = \mathbf{0}$.

d. F. $\|\mathbf{v}\| = \|-\mathbf{v}\|$ for all $\mathbf{v}$ but $\mathbf{v} = -\mathbf{v}$ only holds if $\mathbf{v} = \mathbf{0}$.

f. F. If $t < 0$ they have the *opposite* direction.

h. F. $\|-5\mathbf{v}\| = 5\|\mathbf{v}\|$ for all $\mathbf{v}$, so it fails if $\mathbf{v} \neq \mathbf{0}$.

j. F. Take $\mathbf{w} = -\mathbf{v}$ where $\mathbf{v} \neq \mathbf{0}$.

4.1.22 b. $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$; $x = 3+2t$, $y = -1-t$, $z = 4+5t$

d. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $x = y = z = 1+t$

f. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $x = 2-t$, $y = -1$, $z = 1+t$

4.1.23 b. P corresponds to $t = 2$; Q corresponds to $t = 5$.

4.1.24 b. No intersection

d. $P(2, -1, 3)$; $t = -2$, $s = -3$

4.1.29 $P(3, 1, 0)$ or $P(\frac{5}{3}, \frac{-1}{3}, \frac{4}{3})$

4.1.31 b. $\overrightarrow{CP}_k = -\overrightarrow{CP}_{n+k}$ if $1 \leq k \leq n$, where there are $2n$ points.

4.1.33 $\overrightarrow{DA} = 2\overrightarrow{EA}$ and $2\overrightarrow{AF} = \overrightarrow{FC}$, so $2\overrightarrow{EF} = 2(\overrightarrow{EF}+\overrightarrow{AF}) = \overrightarrow{DA}+\overrightarrow{FC} = \overrightarrow{CB}+\overrightarrow{FC} = \overrightarrow{FC}+\overrightarrow{CB} = \overrightarrow{FB}$. Hence $\overrightarrow{EF} = \frac{1}{2}\overrightarrow{FB}$. So F is the trisection point of both AC and EB.

Section 4.2

4.2.1 b. 6

d. 0

f. 0

4.2.2 b. π or 180°

d. $\frac{\pi}{3}$ or 60°

f. $\frac{2\pi}{3}$ or 120°

4.2.3 b. 1 or -17

4.2.4 b. $t\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

d. $s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

4.2.6 b. $29+57 = 86$

4.2.8 b. $A = B = C = \frac{\pi}{3}$ or 60°

4.2.10 b. $\frac{11}{18}\mathbf{v}$

d. $-\frac{1}{2}\mathbf{v}$

4.2.11 b. $\frac{5}{21}\begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} + \frac{1}{21}\begin{bmatrix} 53 \\ 26 \\ 20 \end{bmatrix}$

d. $\frac{27}{53}\begin{bmatrix} 6 \\ -4 \\ 1 \end{bmatrix} + \frac{1}{53}\begin{bmatrix} -3 \\ 2 \\ 26 \end{bmatrix}$

4.2.12 b. $\frac{1}{26}\sqrt{5642}$, $Q(\frac{71}{26}, \frac{15}{26}, \frac{34}{26})$

4.2.13 b. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} 4 \\ -15 \\ 8 \end{bmatrix}$

4.2.14 b. $-23x+32y+11z=11$

d. $2x-y+z=5$

f. $2x+3y+2z=7$

h. $2x-7y-3z=-1$

j. $x-y-z=3$

4.2.15 b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

f. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}$

4.2.16 b. $\frac{\sqrt{6}}{3}$, $Q(\frac{7}{3}, \frac{2}{3}, \frac{-2}{3})$

4.2.17 b. Yes. The equation is $5x-3y-4z=0$.

4.2.19 b. $(-2, 7, 0)+t(3, -5, 2)$

4.2.20 b. None

d. $P(\frac{13}{19}, \frac{-78}{19}, \frac{65}{19})$

4.2.21 b. $3x+2z=d$, d arbitrary

d. $a(x-3)+b(y-2)+c(z+4)=0$; a, b, and c not all zero

f. $ax+by+(b-a)z=a$; a and b not both zero

h. $ax+by+(a-2b)z=5a-4b$; a and b not both zero

4.2.23 b. $\sqrt{10}$

4.2.24 b. $\frac{\sqrt{14}}{2}$, $A(3, 1, 2)$, $B(\frac{7}{2}, -\frac{1}{2}, 3)$

d. $\frac{\sqrt{6}}{6}$, $A(\frac{19}{3}, 2, \frac{1}{3})$, $B(\frac{37}{6}, \frac{13}{6}, 0)$

4.2.26 b. Consider the diagonal $\mathbf{d} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ The six face diagonals in question are $\pm \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix}, \pm \begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$, $\pm \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix}$. All of these are orthogonal to **d**. The result works for the other diagonals by symmetry.

4.2.28 The four diagonals are (a, b, c), $(-a, b, c)$, $(a, -b, c)$ and $(a, b, -c)$ or their negatives. The dot products are $\pm(-a^2+b^2+c^2)$, $\pm(a^2-b^2+c^2)$, and $\pm(a^2+b^2-c^2)$.

4.2.34 b. The sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the four sides.

4.2.38 b. The angle θ between **u** and $(\mathbf{u}+\mathbf{v}+\mathbf{w})$ is given by $\cos\theta = \frac{\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}+\mathbf{w})}{\|\mathbf{u}\|\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|} = \frac{\|\mathbf{u}\|}{\sqrt{\|\mathbf{u}\|^2+\|\mathbf{v}\|^2+\|\mathbf{w}\|^2}} = \frac{1}{\sqrt{3}}$ because $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\|$. Similar remarks apply to the other angles.

4.2.39 b. Let $\mathbf{p}_0$, $\mathbf{p}_1$ be the vectors of P_0, P_1, so $\mathbf{u} = \mathbf{p}_0 - \mathbf{p}_1$. Then $\mathbf{u}\cdot\mathbf{n} = \mathbf{p}_0\cdot\mathbf{n} - \mathbf{p}_1\cdot\mathbf{n} = (ax_0+by_0)-(ax_1+by_1) = ax_0+by_0+c$. Hence the distance is

$$\left\|\left(\frac{\mathbf{u}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n}\right\| = \frac{|\mathbf{u}\cdot\mathbf{n}|}{\|\mathbf{n}\|}$$

as required.

4.2.41 b. This follows from **(a)** because $\|\mathbf{v}\|^2 = a^2+b^2+c^2$.

4.2.44 d. Take $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$ in **(c)**.

Section 4.3

4.3.3 b. $\pm\frac{\sqrt{3}}{3}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

4.3.4 b. 0

d. $\sqrt{5}$

4.3.5 b. 7

4.3.6 b. The distance is $\|\mathbf{p}-\mathbf{p}_0\|$; use part (a.).

4.3.10 $\|\overrightarrow{AB}\times\overrightarrow{AC}\|$ is the area of the parallelogram determined by A, B, and C.

4.3.12 Because **u** and $\mathbf{v}\times\mathbf{w}$ are parallel, the angle θ between them is 0 or π. Hence $\cos(\theta) = \pm 1$, so the volume is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = \|\mathbf{u}\|\|\mathbf{v}\times\mathbf{w}\|\cos(\theta) = \|\mathbf{u}\|\|(\mathbf{v}\times\mathbf{w})\|$. But the angle between **v** and **w** is $\frac{\pi}{2}$ so $\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\frac{\pi}{2}) = \|\mathbf{v}\|\|\mathbf{w}\|$. The result follows.

4.3.15 b. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$,
then $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \det \begin{bmatrix} \mathbf{i} & u_1 & v_1 + w_1 \\ \mathbf{j} & u_2 & v_2 + w_2 \\ \mathbf{k} & u_3 & v_3 + w_3 \end{bmatrix}$
$= \det \begin{bmatrix} \mathbf{i} & u_1 & v_1 \\ \mathbf{j} & u_2 & v_2 \\ \mathbf{k} & u_3 & v_3 \end{bmatrix} + \det \begin{bmatrix} \mathbf{i} & u_1 & w_1 \\ \mathbf{j} & u_2 & w_2 \\ \mathbf{k} & u_3 & w_3 \end{bmatrix}$
$= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ where we used Exercise 4.3.21.

4.3.16 b. $(\mathbf{v} - \mathbf{w}) \cdot [(\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \times \mathbf{w}) + (\mathbf{w} \times \mathbf{u})] = (\mathbf{v}-\mathbf{w}) \cdot (\mathbf{u} \times \mathbf{v}) + (\mathbf{v}-\mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w}) + (\mathbf{v}-\mathbf{w}) \cdot (\mathbf{w} \times \mathbf{u}) = -\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) + 0 + \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = 0$.

4.3.22 Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be vectors of points in the planes, so $\mathbf{p}_1 \cdot \mathbf{n} = d_1$ and $\mathbf{p}_2 \cdot \mathbf{n} = d_2$. The distance is the length of the projection of $\mathbf{p}_2 - \mathbf{p}_1$ along $\mathbf{n}$; that is $\frac{|(\mathbf{p}_2 - \mathbf{p}_1) \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|d_1 - d_2|}{\|\mathbf{n}\|}$.

Section 4.4

4.4.1 b. $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, projection on $y = -x$.

d. $A = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$, reflection in $y = 2x$.

f. $A = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$, rotation through $\frac{\pi}{3}$.

4.4.2 b. The zero transformation.

4.4.3 b. $\frac{1}{21} \begin{bmatrix} 17 & 2 & -8 \\ 2 & 20 & 4 \\ -8 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

d. $\frac{1}{30} \begin{bmatrix} 22 & -4 & 20 \\ -4 & 28 & 10 \\ 20 & 10 & -20 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

f. $\frac{1}{25} \begin{bmatrix} 9 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$

h. $\frac{1}{11} \begin{bmatrix} -9 & 2 & -6 \\ 2 & -9 & -6 \\ -6 & -6 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$

4.4.4 b. $\frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

4.4.6 $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

4.4.9 a. Write $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$P_L(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\right)\mathbf{d} = \frac{ax+by}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} a^2x + aby \\ abx + b^2y \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 + ab \\ ab + b^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Section 4.5

4.5.1 b. $\frac{1}{2} \begin{bmatrix} \sqrt{2}+2 & 7\sqrt{2}+2 & 3\sqrt{2}+2 & -\sqrt{2}+2 & -5\sqrt{2}+2 \\ -3\sqrt{2}+4 & 3\sqrt{2}+4 & 5\sqrt{2}+4 & \sqrt{2}+4 & 9\sqrt{2}+4 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$

4.5.5 b. $P(\frac{9}{5}, \frac{18}{5})$

Supplementary Exercises for Chapter 4

Supplementary Exercise 4.4. 125 knots in a direction θ degrees east of north, where $\cos\theta = 0.6$ ($\theta = 53°$ or 0.93 radians).

Supplementary Exercise 4.6. $(12, 5)$. Actual speed 12 knots.

Section 5.1

5.1.1 b. Yes

d. No

f. No.

5.1.2 b. No

d. Yes, $\mathbf{x} = 3\mathbf{y} + 4\mathbf{z}$.

5.1.3 b. No

5.1.10 span $\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\} \subseteq$ span $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ by Theorem 5.1.1 because, for each i, $a_i\mathbf{x}_i$ is in span $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$. Similarly, the fact that $\mathbf{x}_i = a_i^{-1}(a_i\mathbf{x}_i)$ is in span $\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\}$ for each i shows that span $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} \subseteq$ span $\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\}$, again by Theorem 5.1.1.

5.1.12 If $\mathbf{y} = r_1\mathbf{x}_1 + \cdots + r_k\mathbf{x}_k$ then $A\mathbf{y} = r_1(A\mathbf{x}_1) + \cdots + r_k(A\mathbf{x}_k) = 0$.

5.1.15 b. $\mathbf{x} = (\mathbf{x} + \mathbf{y}) - \mathbf{y} = (\mathbf{x} + \mathbf{y}) + (-\mathbf{y})$ is in U because U is a subspace and both $\mathbf{x} + \mathbf{y}$ and $-\mathbf{y} = (-1)\mathbf{y}$ are in U.

5.1.16 b. True. $\mathbf{x} = 1\mathbf{x}$ is in U.

d. True. Always $\text{span}\{\mathbf{y},\ \mathbf{z}\} \subseteq \text{span}\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ by Theorem 5.1.1. Since $\mathbf{x}$ is in $\text{span}\{\mathbf{x},\ \mathbf{y}\}$ we have $\text{span}\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\} \subseteq \text{span}\{\mathbf{y},\ \mathbf{z}\}$, again by Theorem 5.1.1.

f. False. $a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} a+2b \\ 0 \end{bmatrix}$ cannot equal $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

5.1.20 If U is a subspace, then S2 and S3 certainly hold. Conversely, assume that S2 and S3 hold for U. Since U is nonempty, choose $\mathbf{x}$ in U. Then $\mathbf{0} = 0\mathbf{x}$ is in U by S3, so S1 also holds. This means that U is a subspace.

5.1.22 b. The zero vector $\mathbf{0}$ is in $U + W$ because $\mathbf{0} = \mathbf{0} + \mathbf{0}$. Let $\mathbf{p}$ and $\mathbf{q}$ be vectors in $U + W$, say $\mathbf{p} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{q} = \mathbf{x}_2 + \mathbf{y}_2$ where $\mathbf{x}_1$ and $\mathbf{x}_2$ are in U, and $\mathbf{y}_1$ and $\mathbf{y}_2$ are in W. Then $\mathbf{p} + \mathbf{q} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2)$ is in $U + W$ because $\mathbf{x}_1 + \mathbf{x}_2$ is in U and $\mathbf{y}_1 + \mathbf{y}_2$ is in W. Similarly, $a(\mathbf{p} + \mathbf{q}) = a\mathbf{p} + a\mathbf{q}$ is in $U + W$ for any scalar a because $a\mathbf{p}$ is in U and $a\mathbf{q}$ is in W. Hence $U + W$ is indeed a subspace of $\mathbb{R}^n$.

Section 5.2

5.2.1 b. Yes. If $r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then $r + s = 0$, $r - s = 0$, and $r + s + t = 0$. These equations give $r = s = t = 0$.

d. No. Indeed:
$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

5.2.2 b. Yes. If $r(\mathbf{x} + \mathbf{y}) + s(\mathbf{y} + \mathbf{z}) + t(\mathbf{z} + \mathbf{x}) = \mathbf{0}$, then $(r + t)\mathbf{x} + (r + s)\mathbf{y} + (s + t)\mathbf{z} = \mathbf{0}$. Since $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent, this implies that $r + t = 0$, $r + s = 0$, and $s + t = 0$. The only solution is $r = s = t = 0$.

d. No. In fact, $(\mathbf{x} + \mathbf{y}) - (\mathbf{y} + \mathbf{z}) + (\mathbf{z} + \mathbf{w}) - (\mathbf{w} + \mathbf{x}) = \mathbf{0}$.

5.2.3 b. $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$; dimension 2.

d. $\left\{\begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}\right\}$; dimension 2.

5.2.4 b. $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$; dimension 2.

d. $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$; dimension 3.

f. $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$; dimension 3.

5.2.5 b. If $r(\mathbf{x} + \mathbf{w}) + s(\mathbf{y} + \mathbf{w}) + t(\mathbf{z} + \mathbf{w}) + u(\mathbf{w}) = \mathbf{0}$, then $r\mathbf{x} + s\mathbf{y} + t\mathbf{z} + (r + s + t + u)\mathbf{w} = \mathbf{0}$, so $r = 0$, $s = 0$, $t = 0$, and $r + s + t + u = 0$. The only solution is $r = s = t = u = 0$, so the set is independent. Since $\dim \mathbb{R}^4 = 4$, the set is a basis by Theorem 5.2.7.

5.2.6 b. Yes

d. Yes

f. No.

5.2.7 b. T. If $r\mathbf{y} + s\mathbf{z} = \mathbf{0}$, then $0\mathbf{x} + r\mathbf{y} + s\mathbf{z} = \mathbf{0}$ so $r = s = 0$ because $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent.

d. F. If $\mathbf{x} \neq \mathbf{0}$, take $k = 2$, $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_2 = -\mathbf{x}$.

f. F. If $\mathbf{y} = -\mathbf{x}$ and $\mathbf{z} = \mathbf{0}$, then $1\mathbf{x} + 1\mathbf{y} + 1\mathbf{z} = \mathbf{0}$.

h. T. This is a nontrivial, vanishing linear combination, so the $\mathbf{x}_i$ cannot be independent.

5.2.10 If $r\mathbf{x}_2 + s\mathbf{x}_3 + t\mathbf{x}_5 = \mathbf{0}$ then $0\mathbf{x}_1 + r\mathbf{x}_2 + s\mathbf{x}_3 + 0\mathbf{x}_4 + t\mathbf{x}_5 + 0\mathbf{x}_6 = \mathbf{0}$ so $r = s = t = 0$.

5.2.12 If $t_1\mathbf{x}_1 + t_2(\mathbf{x}_1 + \mathbf{x}_2) + \cdots + t_k(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k) = \mathbf{0}$, then $(t_1 + t_2 + \cdots + t_k)\mathbf{x}_1 + (t_2 + \cdots + t_k)\mathbf{x}_2 + \cdots + (t_{k-1} + t_k)\mathbf{x}_{k-1} + (t_k)\mathbf{x}_k = \mathbf{0}$. Hence all these coefficients are zero, so we obtain successively $t_k = 0$, $t_{k-1} = 0$, $\dots$, $t_2 = 0$, $t_1 = 0$.

5.2.16 b. We show A^T is invertible (then A is invertible). Let $A^T\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = [s\ t]^T$. This means $as + ct = 0$ and $bs + dt = 0$, so $s(a\mathbf{x} + b\mathbf{y}) + t(c\mathbf{x} + d\mathbf{y}) = (sa + tc)\mathbf{x} + (sb + td)\mathbf{y} = \mathbf{0}$. Hence $s = t = 0$ by hypothesis.

5.2.17 b. Each $V^{-1}\mathbf{x}_i$ is in $\text{null}(AV)$ because $AV(V^{-1}\mathbf{x}_i) = A\mathbf{x}_i = \mathbf{0}$. The set $\{V^{-1}\mathbf{x}_1,\ \dots,\ V^{-1}\mathbf{x}_k\}$ is independent as V^{-1} is invertible. If $\mathbf{y}$ is in $\text{null}(AV)$, then $V\mathbf{y}$ is in $\text{null}(A)$ so let $V\mathbf{y} = t_1\mathbf{x}_1 + \cdots + t_k\mathbf{x}_k$ where each t_k is in $\mathbb{R}$. Thus $\mathbf{y} = t_1V^{-1}\mathbf{x}_1 + \cdots + t_kV^{-1}\mathbf{x}_k$ is in $\text{span}\{V^{-1}\mathbf{x}_1,\ \dots,\ V^{-1}\mathbf{x}_k\}$.

5.2.20 We have $\{\mathbf{0}\} \subseteq U \subseteq W$ where $\dim\{\mathbf{0}\} = 0$ and $\dim W = 1$. Hence $\dim U = 0$ or $\dim U = 1$ by Theorem 5.2.8, that is $U = 0$ or $U = W$, again by Theorem 5.2.8.

Section 5.3

5.3.1 b. $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}, \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}$.

5.3.3 b. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2}(a-c) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{18}(a+4b+c) \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \frac{1}{9}(2a-b+2c) \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

d. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{3}(a+b+c) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}(a-b) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{6}(a+b-2c) \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

5.3.4 b. $\begin{bmatrix} 14 \\ 1 \\ -8 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}$.

5.3.5 b. $t \begin{bmatrix} -1 \\ 3 \\ 10 \\ 11 \end{bmatrix}$, in $\mathbb{R}$

5.3.6 b. $\sqrt{29}$

d. 19

5.3.7 b. F. $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

d. T. Every $\mathbf{x}_i \cdot \mathbf{y}_j = 0$ by assumption, every $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ if $i \neq j$ because the $\mathbf{x}_i$ are orthogonal, and every $\mathbf{y}_i \cdot \mathbf{y}_j = 0$ if $i \neq j$ because the $\mathbf{y}_i$ are orthogonal. As all the vectors are nonzero, this does it.

f. T. Every pair of *distinct* vectors in the set $\{\mathbf{x}\}$ has dot product zero (there are no such pairs).

5.3.9 Let $\mathbf{c}_1, \dots, \mathbf{c}_n$ be the columns of A. Then row i of A^T is $\mathbf{c}_i^T$, so the (i, j)-entry of $A^T A$ is $\mathbf{c}_i^T \mathbf{c}_j = \mathbf{c}_i \cdot \mathbf{c}_j = 0, 1$ according as $i \neq j, i = j$. So $A^T A = I$.

5.3.11 b. Take $n = 3$ in **(a)**, expand, and simplify.

5.3.12 b. We have $(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2$. Hence $(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) = 0$ if and only if $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2$; if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$—where we used the fact that $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{y}\| \geq 0$.

5.3.15 If $A^T A\mathbf{x} = \lambda \mathbf{x}$, then $\|A\mathbf{x}\|^2 = (A\mathbf{x}) \cdot (A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda\|\mathbf{x}\|^2$.

Section 5.4

5.4.1 b. $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}; \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}; 2$

d. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}; \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}; 2$

5.4.2 b. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 6 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

5.4.3 b. No; no

d. No

f. Otherwise, if A is $m \times n$, we have $m = \dim(\text{row } A) = \text{rank } A = \dim(\text{col } A) = n$

5.4.4 Let $A = \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_n \end{bmatrix}$. Then $\text{col } A = \text{span}\{\mathbf{c}_1, \dots, \mathbf{c}_n\} = \{x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n \mid x_i \text{ in } \mathbb{R}\} = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$.

5.4.7 b. The basis is $\left\{ \begin{bmatrix} 6 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ so the dimension is 2.

Have $\text{rank } A = 3$ and $n - 3 = 2$.

5.4.8 b. $n - 1$

5.4.9 b. If $r_1\mathbf{c}_1 + \cdots + r_n\mathbf{c}_n = \mathbf{0}$, let $\mathbf{x} = [r_1, \dots, r_n]^T$. Then $C\mathbf{x} = r_1\mathbf{c}_1 + \cdots + r_n\mathbf{c}_n = \mathbf{0}$, so $\mathbf{x}$ is in $\text{null } A = 0$. Hence each $r_i = 0$.

5.4.10 b. Write $r = \operatorname{rank} A$. Then **(a)** gives $r = \dim(\operatorname{col} A \leq \dim(\operatorname{null} A) = n - r$.

5.4.12 We have $\operatorname{rank}(A) = \dim[\operatorname{col}(A)]$ and $\operatorname{rank}(A^T) = \dim[\operatorname{row}(A^T)]$. Let $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_k\}$ be a basis of $\operatorname{col}(A)$; it suffices to show that $\{\mathbf{c}_1^T, \mathbf{c}_2^T, \dots, \mathbf{c}_k^T\}$ is a basis of $\operatorname{row}(A^T)$. But if $t_1\mathbf{c}_1^T + t_2\mathbf{c}_2^T + \cdots + t_k\mathbf{c}_k^T = \mathbf{0}$, t_j in $\mathbb{R}$, then (taking transposes) $t_1\mathbf{c}_1 + t_2\mathbf{c}_2 + \cdots + t_k\mathbf{c}_k = \mathbf{0}$ so each $t_j = 0$. Hence $\{\mathbf{c}_1^T, \mathbf{c}_2^T, \dots, \mathbf{c}_k^T\}$ is independent. Given $\mathbf{v}$ in $\operatorname{row}(A^T)$ then $\mathbf{v}^T$ is in $\operatorname{col}(A)$; say $\mathbf{v}^T = s_1\mathbf{c}_1 + s_2\mathbf{c}_2 + \cdots + s_k\mathbf{c}_k$, s_j in $\mathbb{R}$: Hence $\mathbf{v} = s_1\mathbf{c}_1^T + s_2\mathbf{c}_2^T + \cdots + s_k\mathbf{c}_k^T$, so $\{\mathbf{c}_1^T, \mathbf{c}_2^T, \dots, \mathbf{c}_k^T\}$ spans $\operatorname{row}(A^T)$, as required.

5.4.15 b. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ be a basis of $\operatorname{col}(A)$. Then $\mathbf{b}$ is *not* in $\operatorname{col}(A)$, so $\{\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{b}\}$ is linearly independent. Show that $\operatorname{col}[A\ \mathbf{b}] = \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{b}\}$.

Section 5.5

5.5.1 b. traces $= 2$, ranks $= 2$, but $\det A = -5$, $\det B = -1$

d. ranks $= 2$, determinants $= 7$, but $\operatorname{tr} A = 5$, $\operatorname{tr} B = 4$

f. traces $= -5$, determinants $= 0$, but $\operatorname{rank} A = 2$, $\operatorname{rank} B = 1$

5.5.3 b. If $B = P^{-1}AP$, then $B^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$.

5.5.4 b. Yes, $P = \begin{bmatrix} -1 & 0 & 6 \\ 0 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

d. No, $c_A(x) = (x+1)(x-4)^2$ so $\lambda = 4$ has multiplicity 2. But $\dim(E_4) = 1$ so Theorem 5.5.6 applies.

5.5.8 b. If $B = P^{-1}AP$ and $A^k = 0$, then $B^k = (P^{-1}AP)^k = P^{-1}A^kP = P^{-1}0P = 0$.

5.5.9 b. The eigenvalues of A are all equal (they are the diagonal elements), so if $P^{-1}AP = D$ is diagonal, then $D = \lambda I$. Hence $A = P^{-1}(\lambda I)P = \lambda I$.

5.5.10 b. A is similar to $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ so (Theorem 5.5.1) $\operatorname{tr} A = \operatorname{tr} D = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

5.5.12 b. $T_P(A)T_P(B) = (P^{-1}AP)(P^{-1}BP) = P^{-1}(AB)P = T_P(AB)$.

5.5.13 b. If A is diagonalizable, so is A^T, and they have the same eigenvalues. Use **(a)**.

5.5.17 b. $c_B(x) = [x - (a+b+c)][x^2 - k]$ where $k = a^2 + b^2 + c^2 - [ab + ac + bc]$. Use Theorem 5.5.7.

Section 5.6

5.6.1 b. $\frac{1}{12}\begin{bmatrix} -20 \\ 46 \\ 95 \end{bmatrix}$, $(A^TA)^{-1}$

$= \frac{1}{12}\begin{bmatrix} 8 & -10 & -18 \\ -10 & 14 & 24 \\ -18 & 24 & 43 \end{bmatrix}$

5.6.2 b. $\frac{64}{13} - \frac{6}{13}x$

d. $-\frac{4}{10} - \frac{17}{10}x$

5.6.3 b. $y = 0.127 - 0.024x + 0.194x^2$, $(M^TM)^{-1} = \frac{1}{4248}\begin{bmatrix} 3348 & 642 & -426 \\ 642 & 571 & -187 \\ -426 & -187 & 91 \end{bmatrix}$

5.6.4 b. $\frac{1}{92}(-46x + 66x^2 + 60 \cdot 2^x)$, $(M^TM)^{-1} = \frac{1}{46}\begin{bmatrix} 115 & 0 & -46 \\ 0 & 17 & -18 \\ -46 & -18 & 38 \end{bmatrix}$

5.6.5 b. $\frac{1}{20}[18 + 21x^2 + 28\sin(\frac{\pi x}{2})]$, $(M^TM)^{-1} = \frac{1}{40}\begin{bmatrix} 24 & -2 & 14 \\ -2 & 1 & 3 \\ 14 & 3 & 49 \end{bmatrix}$

5.6.7 $s = 99.71 - 4.87x$; the estimate of g is 9.74. [The true value of g is 9.81]. If a quadratic in s is fit, the result is $s = 101 - \frac{3}{2}t - \frac{9}{2}t^2$ giving $g = 9$;

$(M^TM)^{-1} = \frac{1}{2}\begin{bmatrix} 38 & -42 & 10 \\ -42 & 49 & -12 \\ 10 & -12 & 3 \end{bmatrix}$.

5.6.9 $y = -5.19 + 0.34x_1 + 0.51x_2 + 0.71x_3$, $(A^TA)^{-1}$

$= \frac{1}{25080}\begin{bmatrix} 517860 & -8016 & 5040 & -22650 \\ -8016 & 208 & -316 & 400 \\ 5040 & -316 & 1300 & -1090 \\ -22650 & 400 & -1090 & 1975 \end{bmatrix}$

5.6.10 b. $f(x) = a_0$ here, so the sum of squares is $S = \sum(y_i - a_0)^2 = na_0^2 - 2a_0\sum y_i + \sum y_i^2$. Completing the square gives $S = n[a_0 - \frac{1}{n}\sum y_i]^2 + [\sum y_i^2 - \frac{1}{n}(\sum y_i)^2]$ This is minimal when $a_0 = \frac{1}{n}\sum y_i$.

5.6.13 b. Here $f(x) = r_0 + r_1e^x$. If $f(x_1) = 0 = f(x_2)$ where $x_1 \neq x_2$, then $r_0 + r_1 \cdot e^{x_1} = 0 = r_0 + r_1 \cdot e^{x_2}$ so $r_1(e^{x_1} - e^{x_2}) = 0$. Hence $r_1 = 0 = r_0$.

Section 5.7

5.7.2 Let X denote the number of years of education, and let Y denote the yearly income (in 1000's). Then $\overline{x} = 15.3$, $s_x^2 = 9.12$ and $s_x = 3.02$, while $\overline{y} = 40.3$, $s_y^2 = 114.23$ and $s_y = 10.69$. The correlation is $r(X, Y) = 0.599$.

5.7.4 b. Given the sample vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, let $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ where $z_i = a + bx_i$ for each i. By **(a)** we have $\overline{z} = a + b\overline{x}$, so

$$\begin{aligned} s_z^2 &= \frac{1}{n-1}\sum_i (z_i - \overline{z})^2 \\ &= \frac{1}{n-1}\sum_i [(a+bx_i) - (a + b\overline{x})]^2 \\ &= \frac{1}{n-1}\sum_i b^2(x_i - \overline{x})^2 \\ &= b^2 s_x^2. \end{aligned}$$

Now **(b)** follows because $\sqrt{b^2} = |b|$.

Supplementary Exercises for Chapter 5

Supplementary Exercise 5.1. b. F

d. T

f. T

h. F

j. F

l. T

n. F

p. F

r. F

Section 6.1

6.1.1 b. No; S5 fails.

d. No; S4 and S5 fail.

6.1.2 b. No; only A1 fails.

d. No.

f. Yes.

h. Yes.

j. No.

l. No; only S3 fails.

n. No; only S4 and S5 fail.

6.1.4 The zero vector is $(0, -1)$; the negative of (x, y) is $(-x, -2-y)$.

6.1.5 b. $\mathbf{x} = \frac{1}{7}(5\mathbf{u} - 2\mathbf{v})$, $\mathbf{y} = \frac{1}{7}(4\mathbf{u} - 3\mathbf{v})$

6.1.6 b. Equating entries gives $a + c = 0$, $b + c = 0$, $b + c = 0$, $a - c = 0$. The solution is $a = b = c = 0$.

d. If $a \sin x + b \cos y + c = 0$ in $\mathbf{F}[0, \pi]$, then this must hold for *every* x in $[0, \pi]$. Taking $x = 0$, $\frac{\pi}{2}$, and π, respectively, gives $b + c = 0$, $a + c = 0$, $-b + c = 0$ whence, $a = b = c = 0$.

6.1.7 b. $4\mathbf{w}$

6.1.10 If $\mathbf{z} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$, then $\mathbf{z} + \mathbf{v} = \mathbf{0} + \mathbf{v}$, so $\mathbf{z} = \mathbf{0}$ by cancellation.

6.1.12 b. $(-a)\mathbf{v} + a\mathbf{v} = (-a + a)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ by Theorem 6.1.3. Because also $-(a\mathbf{v}) + a\mathbf{v} = \mathbf{0}$ (by the definition of $-(a\mathbf{v})$ in axiom A5), this means that $(-a)\mathbf{v} = -(a\mathbf{v})$ by cancellation. Alternatively, use Theorem 6.1.3(4) to give $(-a)\mathbf{v} = [(-1)a]\mathbf{v} = (-1)(a\mathbf{v}) = -(a\mathbf{v})$.

6.1.13 b. The case $n = 1$ is clear, and $n = 2$ is axiom S3. If $n > 2$, then $(a_1 + a_2 + \cdots + a_n)\mathbf{v} = [a_1 + (a_2 + \cdots + a_n)]\mathbf{v} = a_1\mathbf{v} + (a_2 + \cdots + a_n)\mathbf{v} = a_1\mathbf{v} + (a_2\mathbf{v} + \cdots + a_n\mathbf{v})$ using the induction hypothesis; so it holds for all n.

6.1.15 c. If $a\mathbf{v} = a\mathbf{w}$, then $\mathbf{v} = 1\mathbf{v} = (a^{-1}a)\mathbf{v} = a^{-1}(a\mathbf{v}) = a^{-1}(a\mathbf{w}) = (a^{-1}a)\mathbf{w} = 1\mathbf{w} = \mathbf{w}$.

Section 6.2

6.2.1 b. Yes

d. Yes

f. No; not closed under addition or scalar multiplication, and 0 is not in the set.

6.2.2 b. Yes.

d. Yes.

f. No; not closed under addition.

6.2.3 b. No; not closed under addition.

d. No; not closed under scalar multiplication.

f. Yes.

6.2.5 b. If entry k of $\mathbf{x}$ is $x_k \neq 0$, and if $\mathbf{y}$ is in $\mathbb{R}^n$, then $\mathbf{y} = A\mathbf{x}$ where the column of A is $x_k^{-1}\mathbf{y}$, and the other columns are zero.

6.2.6 b. $-3(x+1)+0(x^2+x)+2(x^2+2)$

d. $\frac{2}{3}(x+1)+\frac{1}{3}(x^2+x)-\frac{1}{3}(x^2+2)$

6.2.7 b. No.

d. Yes; $\mathbf{v} = 3\mathbf{u} - \mathbf{w}$.

6.2.8 b. Yes; $1 = \cos^2 x + \sin^2 x$

d. No. If $1 + x^2 = a\cos^2 x + b\sin^2 x$, then taking $x = 0$ and $x = \pi$ gives $a = 1$ and $a = 1 + \pi^2$.

6.2.9 b. Because $\mathbf{P}_2 = \text{span}\{1, x, x^2\}$, it suffices to show that $\{1, x, x^2\} \subseteq \text{span}\{1+2x^2, 3x, 1+x\}$. But $x = \frac{1}{3}(3x); 1 = (1+x) - x$ and $x^2 = \frac{1}{2}[(1+2x^2)-1]$.

6.2.11 b. $\mathbf{u} = (\mathbf{u}+\mathbf{w}) - \mathbf{w}$, $\mathbf{v} = -(\mathbf{u}-\mathbf{v}) + (\mathbf{u}+\mathbf{w}) - \mathbf{w}$, and $\mathbf{w} = \mathbf{w}$

6.2.14 No.

6.2.17 b. Yes.

6.2.18 $\mathbf{v}_1 = \frac{1}{a_1}\mathbf{u} - \frac{a_2}{a_1}\mathbf{v}_2 - \cdots - \frac{a_n}{a_1}\mathbf{v}_n$, so $V \subseteq \text{span}\{\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_n\}$

6.2.21 b. $\mathbf{v} = (\mathbf{u}+\mathbf{v}) - \mathbf{u}$ is in U.

6.2.22 Given the condition and $\mathbf{u} \in U$, $\mathbf{0} = \mathbf{u} + (-1)\mathbf{u} \in U$. The converse holds by the subspace test.

Section 6.3

6.3.1 b. If $ax^2 + b(x+1) + c(1-x-x^2) = 0$, then $a+c=0$, $b-c=0$, $b+c=0$, so $a=b=c=0$.

d. If $a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $a+c+d=0$, $a+b+d=0$, $a+b+c=0$, and $b+c+d=0$, so $a=b=c=d=0$.

6.3.2 b. $3(x^2-x+3) - 2(2x^2+x+5) + (x^2+5x+1) = 0$

d. $2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

f. $\frac{5}{x^2+x-6} + \frac{1}{x^2-5x+6} - \frac{6}{x^2-9} = 0$

6.3.3 b. Dependent: $1 - \sin^2 x - \cos^2 x = 0$

6.3.4 b. $x \neq -\frac{1}{3}$

6.3.5 b. If $r(-1, 1, 1) + s(1, -1, 1) + t(1, 1, -1) = (0, 0, 0)$, then $-r+s+t=0$, $r-s+t=0$, and $r-s-t=0$, and this implies that $r=s=t=0$. This proves independence. To prove that they span $\mathbb{R}^3$, observe that $(0, 0, 1) = \frac{1}{2}[(-1, 1, 1) + (1, -1, 1)]$ so $(0, 0, 1)$ lies in $\text{span}\{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$. The proof is similar for $(0, 1, 0)$ and $(1, 0, 0)$.

d. If $r(1+x) + s(x+x^2) + t(x^2+x^3) + ux^3 = 0$, then $r=0$, $r+s=0$, $s+t=0$, and $t+u=0$, so $r=s=t=u=0$. This proves independence. To show that they span $\mathbf{P}_3$, observe that $x^2 = (x^2+x^3) - x^3$, $x = (x+x^2) - x^2$, and $1 = (1+x) - x$, so $\{1, x, x^2, x^3\} \subseteq \text{span}\{1+x, x+x^2, x^2+x^3, x^3\}$.

6.3.6 b. $\{1, x+x^2\}$; dimension $= 2$

d. $\{1, x^2\}$; dimension $= 2$

6.3.7 b. $\left\{\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$; dimension $= 2$

d. $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$; dimension $= 2$

6.3.8 b. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

6.3.10 b. $\dim V = 7$

6.3.11 b. $\{x^2-x, x(x^2-x), x^2(x^2-x), x^3(x^2-x)\}$; $\dim V = 4$

6.3.12 b. No. Any linear combination f of such polynomials has $f(0) = 0$.

d. No. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$; consists of invertible matrices.

f. Yes. $0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w} = \mathbf{0}$ for every set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

h. Yes. $s\mathbf{u} + t(\mathbf{u}+\mathbf{v}) = \mathbf{0}$ gives $(s+t)\mathbf{u} + t\mathbf{v} = \mathbf{0}$, whence $s+t = 0 = t$.

j. Yes. If $r\mathbf{u} + s\mathbf{v} = \mathbf{0}$, then $r\mathbf{u} + s\mathbf{v} + 0\mathbf{w} = \mathbf{0}$, so $r = 0 = s$.

l. Yes. $\mathbf{u}+\mathbf{v}+\mathbf{w} \neq \mathbf{0}$ because $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent.

n. Yes. If I is independent, then $|I| \leq n$ by the fundamental theorem because any basis spans V.

6.3.15 If a linear combination of the subset vanishes, it is a linear combination of the vectors in the larger set (coefficients outside the subset are zero) so it is trivial.

6.3.19 Because $\{\mathbf{u},\ \mathbf{v}\}$ is linearly independent, $s\mathbf{u}'+t\mathbf{v}'=\mathbf{0}$ is equivalent to $\left[\begin{array}{cc} a & c \\ b & d \end{array}\right]\left[\begin{array}{c} s \\ t \end{array}\right]=\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$. Now apply Theorem 2.4.5.

6.3.23 b. Independent.

d. Dependent. For example, $(\mathbf{u}+\mathbf{v})-(\mathbf{v}+\mathbf{w})+(\mathbf{w}+\mathbf{z})-(\mathbf{z}+\mathbf{u})=\mathbf{0}$.

6.3.26 If z is not real and $az+bz^2=0$, then $a+bz=0 (z \neq 0)$. Hence if $b \neq 0$, then $z=-ab^{-1}$ is real. So $b=0$, and so $a=0$. Conversely, if z is real, say $z=a$, then $(-a)z+1z^2=0$, contrary to the independence of $\{z,\ z^2\}$.

6.3.29 b. If $U\mathbf{x}=\mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, then $R\mathbf{x}=\mathbf{0}$ where $R \neq 0$ is row 1 of U. If $B \in \mathbf{M}_{mn}$ has each row equal to R, then $B\mathbf{x} \neq \mathbf{0}$. But if $B=\sum r_i A_i U$, then $B\mathbf{x}=\sum r_i A_i U\mathbf{x}=\mathbf{0}$. So $\{A_i U\}$ cannot span $\mathbf{M}_{mn}$.

6.3.33 b. If $U \cap W=0$ and $r\mathbf{u}+s\mathbf{w}=\mathbf{0}$, then $r\mathbf{u}=-s\mathbf{w}$ is in $U \cap W$, so $r\mathbf{u}=\mathbf{0}=s\mathbf{w}$. Hence $r=0=s$ because $\mathbf{u} \neq \mathbf{0} \neq \mathbf{w}$. Conversely, if $\mathbf{v} \neq \mathbf{0}$ lies in $U \cap W$, then $1\mathbf{v}+(-1)\mathbf{v}=\mathbf{0}$, contrary to hypothesis.

6.3.36 b. $\dim O_n=\frac{n}{2}$ if n is even and $\dim O_n=\frac{n+1}{2}$ if n is odd.

Section 6.4

6.4.1 b. $\{(0,\ 1,\ 1),\ (1,\ 0,\ 0),\ (0,\ 1,\ 0)\}$

d. $\{x^2-x+1,\ 1,\ x\}$

6.4.2 b. Any three except $\{x^2+3,\ x+2,\ x^2-2x-1\}$

6.4.3 b. Add $(0,\ 1,\ 0,\ 0)$ and $(0,\ 0,\ 1,\ 0)$.

d. Add 1 and x^3.

6.4.4 b. If $z=a+bi$, then $a \neq 0$ and $b \neq 0$. If $rz+s\bar{z}=0$, then $(r+s)a=0$ and $(r-s)b=0$. This means that $r+s=0=r-s$, so $r=s=0$. Thus $\{z,\ \bar{z}\}$ is independent; it is a basis because $\dim \mathbb{C}=2$.

6.4.5 b. The polynomials in S have distinct degrees.

6.4.6 b. $\{4,\ 4x,\ 4x^2,\ 4x^3\}$ is one such basis of $\mathbf{P}_3$. However, there is *no* basis of $\mathbf{P}_3$ consisting of polynomials that have the property that their coefficients sum to zero. For if such a basis exists, then every polynomial in $\mathbf{P}_3$ would have this property (because sums and scalar multiples of such polynomials have the same property).

6.4.7 b. Not a basis.

d. Not a basis.

6.4.8 b. Yes; no.

6.4.10 $\det A=0$ if and only if A is not invertible; if and only if the rows of A are dependent (Theorem 5.2.3); if and only if some row is a linear combination of the others (Lemma 6.4.2).

6.4.11 b. No. $\{(0,\ 1),\ (1,\ 0)\} \subseteq \{(0,\ 1),\ (1,\ 0),\ (1,\ 1)\}$.

d. Yes. See Exercise 6.3.15.

6.4.15 If $\mathbf{v} \in U$ then $W=U$; if $\mathbf{v} \notin U$ then $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k,\ \mathbf{v}\}$ is a basis of W by the independent lemma.

6.4.18 b. Two distinct planes through the origin (U and W) meet in a line through the origin ($U \cap W$).

6.4.23 b. The set $\{(1,\ 0,\ 0,\ 0,\ \dots),\ (0,\ 1,\ 0,\ 0,\ 0,\ \dots),\ (0,\ 0,\ 1,\ 0,\ 0,\ \dots),\ \dots\}$ contains independent subsets of arbitrary size.

6.4.25 b. $\mathbb{R}\mathbf{u}+\mathbb{R}\mathbf{w}=\{r\mathbf{u}+s\mathbf{w} \mid r,\ s \text{ in } \mathbb{R}\}=\operatorname{span}\{\mathbf{u},\ \mathbf{w}\}$

Section 6.5

6.5.2 b. $3+4(x-1)+3(x-1)^2+(x-1)^3$

d. $1+(x-1)^3$

6.5.6 b. The polynomials are $(x-1)(x-2)$, $(x-1)(x-3)$, $(x-2)(x-3)$. Use $a_0=3$, $a_1=2$, and $a_2=1$.

6.5.7 b. $f(x)=\frac{3}{2}(x-2)(x-3)-7(x-1)(x-3)+\frac{13}{2}(x-1)(x-2)$.

6.5.10 b. If $r(x-a)^2+s(x-a)(x-b)+t(x-b)^2=0$, then evaluation at $x=a (x=b)$ gives $t=0 (r=0)$. Thus $s(x-a)(x-b)=0$, so $s=0$. Use Theorem 6.4.4.

6.5.11 b. Suppose $\{p_0(x),\ p_1(x),\ \dots,\ p_{n-2}(x)\}$ is a basis of $\mathbf{P}_{n-2}$. We show that $\{(x-a)(x-b)p_0(x),\ (x-a)(x-b)p_1(x),\ \dots,\ (x-a)(x-b)p_{n-2}(x)\}$ is a basis of U_n. It is a spanning set by part **(a)**, so assume that a linear combination vanishes with coefficients $r_0,\ r_1,\ \dots,\ r_{n-2}$. Then $(x-a)(x-b)[r_0p_0(x)+\cdots+r_{n-2}p_{n-2}(x)]=0$, so $r_0p_0(x)+\cdots+r_{n-2}p_{n-2}(x)=0$ by the Hint. This implies that $r_0=\cdots=r_{n-2}=0$.

Section 6.6

6.6.1 b. e^{1-x}

d. $\frac{e^{2x}-e^{-3x}}{e^2-e^{-3}}$

f. $2e^{2x}(1+x)$

h. $\frac{e^{ax}-e^{a(2-x)}}{1-e^{2a}}$

j. $e^{\pi-2x}\sin x$

6.6.4 b. $ce^{-x}+2$, c a constant

6.6.5 b. $ce^{-3x}+de^{2x}-\frac{x^3}{3}$

6.6.6 b. $t=\frac{3\ln(\frac{1}{2})}{\ln(\frac{4}{5})}=9.32$ hours

6.6.8 $k=(\frac{\pi}{15})^2=0.044$

Supplementary Exercises for Chapter 6

Supplementary Exercise 6.2. b. If $YA=0$, Y a row, we show that $Y=0$; thus A^T (and hence A) is invertible. Given a column $\mathbf{c}$ in $\mathbb{R}^n$ write $\mathbf{c}=\sum_i r_i(A\mathbf{v}_i)$ where each r_i is in $\mathbb{R}$. Then $Y\mathbf{c}=\sum_i r_iYA\mathbf{v}_i$, so

$Y=YI_n=Y\left[\begin{array}{cccc}\mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{array}\right]=\left[\begin{array}{cccc}Y\mathbf{e}_1 & Y\mathbf{e}_2 & \cdots & Y\mathbf{e}_n\end{array}\right]=\left[\begin{array}{cccc}0 & 0 & \cdots & 0\end{array}\right]=0$, as required.

Supplementary Exercise 6.4. We have $\operatorname{null} A\subseteq\operatorname{null}(A^TA)$ because $A\mathbf{x}=\mathbf{0}$ implies $(A^TA)\mathbf{x}=\mathbf{0}$. Conversely, if $(A^TA)\mathbf{x}=\mathbf{0}$, then $\|A\mathbf{x}\|^2=(A\mathbf{x})^T(A\mathbf{x})=\mathbf{x}^TA^TA\mathbf{x}=0$. Thus $A\mathbf{x}=\mathbf{0}$.

Section 7.1

7.1.1 b. $T(\mathbf{v})=\mathbf{v}A$ where $A=\left[\begin{array}{rrr}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{array}\right]$

d. $T(A+B)=P(A+B)Q=PAQ+PBQ=T(A)+T(B)$; $T(rA)=P(rA)Q=rPAQ=rT(A)$

f. $T[(p+q)(x)]=(p+q)(0)=p(0)+q(0)=T[p(x)]+T[q(x)]$;
$T[(rp)(x)]=(rp)(0)=r(p(0))=rT[p(x)]$

h. $T(X+Y)=(X+Y)\cdot Z=X\cdot Z+Y\cdot Z=T(X)+T(Y)$, and $T(rX)=(rX)\cdot Z=r(X\cdot Z)=rT(X)$

j. If $\mathbf{v}=(v_1,\ \dots,\ v_n)$ and $\mathbf{w}=(w_1,\ \dots,\ w_n)$, then $T(\mathbf{v}+\mathbf{w})=(v_1+w_1)\mathbf{e}_1+\cdots+(v_n+w_n)\mathbf{e}_n=(v_1\mathbf{e}_1+\cdots+v_n\mathbf{e}_n)+(w_1\mathbf{e}_1+\cdots+w_n\mathbf{e}_n)=T(\mathbf{v})+T(\mathbf{w})$
$T(a\mathbf{v})=(av_1)\mathbf{e}+\cdots+(av_n)\mathbf{e}_n=a(v\mathbf{e}+\cdots+v_n\mathbf{e}_n)=aT(\mathbf{v})$

7.1.2 b. $\operatorname{rank}(A+B)\neq\operatorname{rank}A+\operatorname{rank}B$ in general. For example, $A=\left[\begin{array}{rr}1 & 0\\ 0 & 1\end{array}\right]$ and $B=\left[\begin{array}{rr}1 & 0\\ 0 & -1\end{array}\right]$.

d. $T(\mathbf{0})=\mathbf{0}+\mathbf{u}=\mathbf{u}\neq\mathbf{0}$, so T is not linear by Theorem 7.1.1.

7.1.3 b. $T(3\mathbf{v}_1+2\mathbf{v}_2)=0$

d. $T\left[\begin{array}{r}1\\ -7\end{array}\right]=\left[\begin{array}{r}-3\\ 4\end{array}\right]$

f. $T(2-x+3x^2)=46$

7.1.4 b. $T(x,\ y)=\frac{1}{3}(x-y,\ 3y,\ x-y)$; $T(-1,\ 2)=(-1,\ 2,\ -1)$

d. $T\left[\begin{array}{cc}a & b\\ c & d\end{array}\right]=3a-3c+2b$

7.1.5 b. $T(\mathbf{v})=\frac{1}{3}(7\mathbf{v}-9\mathbf{w})$, $T(\mathbf{w})=\frac{1}{3}(\mathbf{v}+3\mathbf{w})$

7.1.8 b. $T(\mathbf{v})=(-1)\mathbf{v}$ for all $\mathbf{v}$ in V, so T is the scalar operator -1.

7.1.12 If $T(1)=\mathbf{v}$, then $T(r)=T(r\cdot 1)=rT(1)=r\mathbf{v}$ for all r in $\mathbb{R}$.

7.1.15 b. $\mathbf{0}$ is in $U=\{\mathbf{v}\in V\mid T(\mathbf{v})\in P\}$ because $T(\mathbf{0})=\mathbf{0}$ is in P. If $\mathbf{v}$ and $\mathbf{w}$ are in U, then $T(\mathbf{v})$ and $T(\mathbf{w})$ are in P. Hence $T(\mathbf{v}+\mathbf{w})=T(\mathbf{v})+T(\mathbf{w})$ is in P and $T(r\mathbf{v})=rT(\mathbf{v})$ is in P, so $\mathbf{v}+\mathbf{w}$ and $r\mathbf{v}$ are in U.

7.1.18 Suppose $r\mathbf{v}+sT(\mathbf{v})=\mathbf{0}$. If $s=0$, then $r=0$ (because $\mathbf{v}\neq\mathbf{0}$). If $s\neq 0$, then $T(\mathbf{v})=a\mathbf{v}$ where $a=-s^{-1}r$. Thus $\mathbf{v}=T^2(\mathbf{v})=T(a\mathbf{v})=a^2\mathbf{v}$, so $a^2=1$, again because $\mathbf{v}\neq\mathbf{0}$. Hence $a=\pm 1$. Conversely, if $T(\mathbf{v})=\pm\mathbf{v}$, then $\{\mathbf{v},\ T(\mathbf{v})\}$ is certainly not independent.

7.1.21 b. Given such a T, write $T(x)=a$. If $p=p(x)=\sum_{i=0}^n a_ix^i$, then $T(p)=\sum a_iT(x^i)=\sum a_i[T(x)]^i=\sum a_ia^i=p(a)=E_a(p)$. Hence $T=E_a$.

Section 7.2

7.2.1 b.
$$\left\{\begin{bmatrix}-3\\7\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}\right\}; \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}; 2, 2$$

d. $\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}\right\}; \left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\-2\end{bmatrix}\right\}; 2, 1$

7.2.2 b. $\{x^2-x\}$; $\{(1, 0), (0, 1)\}$

d. $\{(0, 0, 1)\}$; $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$

f. $\left\{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}; \{1\}$

h. $\{(1, 0, 0, \dots, 0, -1), (0, 1, 0, \dots, 0, -1), \dots, (0, 0, 0, \dots, 1, -1)\}$; $\{1\}$

j. $\left\{\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$; $\left\{\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\right\}$

7.2.3 b. $T(\mathbf{v}) = \mathbf{0} = (0, 0)$ if and only if $P(\mathbf{v}) = 0$ and $Q(\mathbf{v}) = 0$; that is, if and only if $\mathbf{v}$ is in $\ker P \cap \ker Q$.

7.2.4 b. $\ker T = \text{span}\{(-4, 1, 3)\}$; $B = \{(1, 0, 0), (0, 1, 0), (-4, 1, 3)\}$, $\text{im } T = \text{span}\{(1, 2, 0, 3), (1, -1, -3, 0)\}$

7.2.6 b. Yes. $\dim(\text{im } T) = 5 - \dim(\ker T) = 3$, so $\text{im } T = W$ as $\dim W = 3$.

d. No. $T = 0 : \mathbb{R}^2 \to \mathbb{R}^2$

f. No. $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (y, 0)$. Then $\ker T = \text{im } T$

h. Yes. $\dim V = \dim(\ker T) + \dim(\text{im } T) \leq \dim W + \dim W = 2 \dim W$

j. No. Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (y, 0)$.

l. No. Same example as **(j)**.

n. No. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x, 0)$. If $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$, then $\mathbb{R}^2 = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ but $\mathbb{R}^2 \neq \text{span}\{T(\mathbf{v}_1), T(\mathbf{v}_2)\}$.

7.2.7 b. Given $\mathbf{w}$ in W, let $\mathbf{w} = T(\mathbf{v})$, $\mathbf{v}$ in V, and write $\mathbf{v} = r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n$. Then $\mathbf{w} = T(\mathbf{v}) = r_1T(\mathbf{v}_1) + \cdots + r_nT(\mathbf{v}_n)$.

7.2.8 b. $\text{im } T = \{\sum_i r_i\mathbf{v}_i \mid r_i \text{ in } \mathbb{R}\} = \text{span}\{\mathbf{v}_i\}$.

7.2.10 T is linear and onto. Hence $1 = \dim \mathbb{R} = \dim(\text{im } T) = \dim(\mathbf{M}_{nn}) - \dim(\ker T) = n^2 - \dim(\ker T)$.

7.2.12 The condition means $\ker(T_A) \subseteq \ker(T_B)$, so $\dim[\ker(T_A)] \leq \dim[\ker(T_B)]$. Then Theorem 7.2.4 gives $\dim[\text{im}(T_A)] \geq \dim[\text{im}(T_B)]$; that is, $\text{rank } A \geq \text{rank } B$.

7.2.15 b. $B = \{x-1, \dots, x^n-1\}$ is independent (distinct degrees) and contained in $\ker T$. Hence B is a basis of $\ker T$ by **(a)**.

7.2.20 Define $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ by $T(A) = A - A^T$ for all A in $\mathbf{M}_{nn}$. Then $\ker T = U$ and $\text{im } T = V$ by Example 7.2.3, so the dimension theorem gives $n^2 = \dim \mathbf{M}_{nn} = \dim(U) + \dim(V)$.

7.2.22 Define $T : \mathbf{M}_{nn} \to \mathbb{R}^n$ by $T(A) = A\mathbf{y}$ for all A in $\mathbf{M}_{nn}$. Then T is linear with $\ker T = U$, so it is enough to show that T is onto (then $\dim U = n^2 - \dim(\text{im } T) = n^2 - n$). We have $T(0) = \mathbf{0}$. Let $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T \neq \mathbf{0}$ in $\mathbb{R}^n$. If $y_k \neq \mathbf{0}$ let $\mathbf{c}_k = y_k^{-1}\mathbf{y}$, and let $\mathbf{c}_j = \mathbf{0}$ if $j \neq k$. If $A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}$, then $T(A) = A\mathbf{y} = y_1\mathbf{c}_1 + \cdots + y_k\mathbf{c}_k + \cdots + y_n\mathbf{c}_n = \mathbf{y}$. This shows that T is onto, as required.

7.2.29 b. By Lemma 6.4.2, let $\{\mathbf{u}_1, \dots, \mathbf{u}_m, \dots, \mathbf{u}_n\}$ be a basis of V where $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a basis of U. By Theorem 7.1.3 there is a linear transformation $S : V \to V$ such that $S(\mathbf{u}_i) = \mathbf{u}_i$ for $1 \leq i \leq m$, and $S(\mathbf{u}_i) = \mathbf{0}$ if $i > m$. Because each $\mathbf{u}_i$ is in $\text{im } S$, $U \subseteq \text{im } S$. But if $S(\mathbf{v})$ is in $\text{im } S$, write $\mathbf{v} = r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m + \cdots + r_n\mathbf{u}_n$. Then $S(\mathbf{v}) = r_1S(\mathbf{u}_1) + \cdots + r_mS(\mathbf{u}_m) = r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m$ is in U. So $\text{im } S \subseteq U$.

Section 7.3

7.3.1 b. T is onto because $T(1, -1, 0) = (1, 0, 0)$, $T(0, 1, -1) = (0, 1, 0)$, and $T(0, 0, 1) = (0, 0, 1)$. Use Theorem 7.3.3.

d. T is one-to-one because $0 = T(X) = UXV$ implies that $X = 0$ (U and V are invertible). Use Theorem 7.3.3.

f. T is one-to-one because $\mathbf{0} = T(\mathbf{v}) = k\mathbf{v}$ implies that $\mathbf{v} = \mathbf{0}$ (because $k \neq 0$). T is onto because $T\left(\frac{1}{k}\mathbf{v}\right) = \mathbf{v}$ for all $\mathbf{v}$. [Here Theorem 7.3.3 does not apply if $\dim V$ is not finite.]

h. T is one-to-one because $T(A) = 0$ implies $A^T = 0$, whence $A = 0$. Use Theorem 7.3.3.

7.3.4 b. $ST(x, y, z) = (x+y, 0, y+z)$, $TS(x, y, z) = (x, 0, z)$

d. $ST\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}c & 0\\0 & d\end{bmatrix}$, $TS\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}0 & a\\d & 0\end{bmatrix}$

7.3.5 b. $T^2(x, y) = T(x+y, 0) = (x+y, 0) = T(x, y)$. Hence $T^2 = T$.

d. $T^2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}T\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$

7.3.6 b. No inverse; $(1, -1, 1, -1)$ is in $\ker T$.

d. $T^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3a-2c & 3b-2d \\ a+c & b+d \end{bmatrix}$

f. $T^{-1}(a, b, c) = \frac{1}{2}\left[2a+(b-c)x-(2a-b-c)x^2\right]$

7.3.7 b. $T^2(x, y) = T(ky-x, y) = (ky-(ky-x), y) = (x, y)$

d. $T^2(X) = A^2X = IX = X$

7.3.8 b. $T^3(x, y, z, w) = (x, y, z, -w)$ so $T^6(x, y, z, w) = T^3\left[T^3(x, y, z, w)\right] = (x, y, z, w)$. Hence $T^{-1} = T^5$. So $T^{-1}(x, y, z, w) = (y-x, -x, z, -w)$.

7.3.9 b. $T^{-1}(A) = U^{-1}A$.

7.3.10 b. Given $\mathbf{u}$ in U, write $\mathbf{u} = S(\mathbf{w})$, $\mathbf{w}$ in W (because S is onto). Then write $\mathbf{w} = T(\mathbf{v})$, $\mathbf{v}$ in V (T is onto). Hence $\mathbf{u} = ST(\mathbf{v})$, so ST is onto.

7.3.12 b. For all $\mathbf{v}$ in V, $(RT)(\mathbf{v}) = R[T(\mathbf{v})]$ is in $\operatorname{im}(R)$.

7.3.13 b. Given $\mathbf{w}$ in W, write $\mathbf{w} = ST(\mathbf{v})$, $\mathbf{v}$ in V (ST is onto). Then $\mathbf{w} = S[T(\mathbf{v})]$, $T(\mathbf{v})$ in U, so S is onto. But then $\operatorname{im} S = W$, so $\dim U = \dim(\ker S) + \dim(\operatorname{im} S) \geq \dim(\operatorname{im} S) = \dim W$.

7.3.16 $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T$ by Theorem 7.2.5. So $T : \operatorname{span}\{\mathbf{e}_1, \dots, \mathbf{e}_r\} \to \operatorname{im} T$ is an isomorphism by Theorem 7.3.1.

7.3.19 b. $T(x, y) = (x, y+1)$

7.3.24 b. $TS[x_0, x_1, \dots) = T[0, x_0, x_1, \dots) = [x_0, x_1, \dots)$, so $TS = 1_V$. Hence TS is both onto and one-to-one, so T is onto and S is one-to-one by Exercise 7.3.13. But $[1, 0, 0, \dots)$ is in $\ker T$ while $[1, 0, 0, \dots)$ is not in $\operatorname{im} S$.

7.3.26 b. If $T(p) = 0$, then $p(x) = -xp'(x)$. We write $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, and this becomes $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = -a_1x - 2a_2x^2 - \cdots - na_nx^n$. Equating coefficients yields $a_0 = 0$, $2a_1 = 0$, $3a_2 = 0$, $\dots$, $(n+1)a_n = 0$, whence $p(x) = 0$. This means that $\ker T = 0$, so T is one-to-one. But then T is an isomorphism by Theorem 7.3.3.

7.3.27 b. If $ST = 1_V$ for some S, then T is onto by Exercise 7.3.13. If T is onto, let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$. Since T is onto, $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T = W$ by Theorem 7.2.5. Thus $S : W \to V$ is an isomorphism where by $S\{T(\mathbf{e}_i)] = \mathbf{e}_i$ for $i = 1, 2, \dots, r$. Hence $TS[T(\mathbf{e}_i)] = T(\mathbf{e}_i)$ for each i, that is $TS[T(\mathbf{e}_i)] = 1_W[T(\mathbf{e}_i)]$. This means that $TS = 1_W$ because they agree on the basis $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ of W.

7.3.28 b. If $T = SR$, then every vector $T(\mathbf{v})$ in $\operatorname{im} T$ has the form $T(\mathbf{v}) = S[R(\mathbf{v})]$, whence $\operatorname{im} T \subseteq \operatorname{im} S$. Since R is invertible, $S = TR^{-1}$ implies $\operatorname{im} S \subseteq \operatorname{im} T$.

Conversely, assume that $\operatorname{im} S = \operatorname{im} T$. Then $\dim(\ker S) = \dim(\ker T)$ by the dimension theorem. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_r, \mathbf{f}_{r+1}, \dots, \mathbf{f}_n\}$ be bases of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_{r+1}, \dots, \mathbf{f}_n\}$ are bases of $\ker S$ and $\ker T$, respectively. By Theorem 7.2.5, $\{S(\mathbf{e}_1), \dots, S(\mathbf{e}_r)\}$ and $\{T(\mathbf{f}_1), \dots, T(\mathbf{f}_r)\}$ are both bases of $\operatorname{im} S = \operatorname{im} T$. So let $\mathbf{g}_1, \dots, \mathbf{g}_r$ in V be such that $S(\mathbf{e}_i) = T(\mathbf{g}_i)$ for each $i = 1, 2, \dots, r$. Show that

$$B = \{\mathbf{g}_1, \dots, \mathbf{g}_r, \mathbf{f}_{r+1}, \dots, \mathbf{f}_n\} \text{ is a basis of } V.$$

Then define $R : V \to V$ by $R(\mathbf{g}_i) = \mathbf{e}_i$ for $i = 1, 2, \dots, r$, and $R(\mathbf{f}_j) = \mathbf{e}_j$ for $j = r+1, \dots, n$. Then R is an isomorphism by Theorem 7.3.1. Finally $SR = T$ since they have the same effect on the basis B.

7.3.29 Let $B = \{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ be a basis of V with $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ a basis of $\ker T$. If $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r), \mathbf{w}_{r+1}, \dots, \mathbf{w}_n\}$ is a basis of V, define S by $S[T(\mathbf{e}_i)] = \mathbf{e}_i$ for $1 \leq i \leq r$, and $S(\mathbf{w}_j) = \mathbf{e}_j$ for $r+1 \leq j \leq n$. Then S is an isomorphism by Theorem 7.3.1, and $TST(\mathbf{e}_i) = T(\mathbf{e}_i)$ clearly holds for $1 \leq i \leq r$. But if $i \geq r+1$, then $T(\mathbf{e}_i) = \mathbf{0} = TST(\mathbf{e}_i)$, so $T = TST$ by Theorem 7.1.2.

Section 7.5

7.5.1 b. $\{[1), [2^n), [(-3)^n)\}$; $x_n = \frac{1}{20}(15 + 2^{n+3} + (-3)^{n+1})$

7.5.2 b. $\{[1), [n), [(-2)^n)\}$; $x_n = \frac{1}{9}(5 - 6n + (-2)^{n+2})$

d. $\{[1), [n), [n^2)\}$; $x_n = 2(n-1)^2 - 1$

7.5.3 b. $\{[a^n), [b^n)\}$

7.5.4 b. $[1, 0, 0, 0, 0, \ldots), [0, 1, 0, 0, 0, \ldots), [0, 0, 1, 1, 1, \ldots), [0, 0, 1, 2, 3, \ldots)$

7.5.7 By Remark 2,

$$[i^n + (-i)^n) = [2, 0, -2, 0, 2, 0, -2, 0, \ldots)$$
$$[i(i^n - (-i)^n)) = [0, -2, 0, 2, 0, -2, 0, 2, \ldots)$$

are solutions. They are linearly independent and so are a basis.

Section 8.1

8.1.1 b. $\{(2, 1), \frac{3}{5}(-1, 2)\}$

d. $\{(0, 1, 1), (1, 0, 0), (0, -2, 2)\}$

8.1.2 b. $\mathbf{x} = \frac{1}{182}(271, -221, 1030) + \frac{1}{182}(93, 403, 62)$

d. $\mathbf{x} = \frac{1}{4}(1, 7, 11, 17) + \frac{1}{4}(7, -7, -7, 7)$

f. $\mathbf{x} = \frac{1}{12}(5a-5b+c-3d, -5a+5b-c+3d, a-b+11c+3d, -3a+3b+3c+3d) + \frac{1}{12}(7a+5b-c+3d, 5a+7b+c-3d, -a+b+c-3d, 3a-3b-3c+9d)$

8.1.3 a. $\frac{1}{10}(-9, 3, -21, 33) = \frac{3}{10}(-3, 1, -7, 11)$

c. $\frac{1}{70}(-63, 21, -147, 231) = \frac{3}{10}(-3, 1, -7, 11)$

8.1.4 b. $\{(1, -1, 0), \frac{1}{2}(-1, -1, 2)\}$; $\text{proj}_U\, \mathbf{x} = (1, 0, -1)$

d. $\{(1, -1, 0, 1), (1, 1, 0, 0), \frac{1}{3}(-1, 1, 0, 2)\}$; $\text{proj}_U\, \mathbf{x} = (2, 0, 0, 1)$

8.1.5 b. $U^\perp = \text{span}\,\{(1, 3, 1, 0), (-1, 0, 0, 1)\}$

8.1.8 Write $\mathbf{p} = \text{proj}_U\, \mathbf{x}$. Then $\mathbf{p}$ is in U by definition. If $\mathbf{x}$ is U, then $\mathbf{x}-\mathbf{p}$ is in U. But $\mathbf{x}-\mathbf{p}$ is also in $U^\perp$ by Theorem 8.1.3, so $\mathbf{x}-\mathbf{p}$ is in $U \cap U^\perp = \{\mathbf{0}\}$. Thus $\mathbf{x} = \mathbf{p}$.

8.1.10 Let $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ be an orthonormal basis of U. If $\mathbf{x}$ is in U the expansion theorem gives $\mathbf{x} = (\mathbf{x}\cdot\mathbf{f}_1)\mathbf{f}_1 + (\mathbf{x}\cdot\mathbf{f}_2)\mathbf{f}_2 + \cdots + (\mathbf{x}\cdot\mathbf{f}_m)\mathbf{f}_m = \text{proj}_U\, \mathbf{x}$.

8.1.14 Let $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m\}$ be a basis of $U^\perp$, and let A be the $n\times n$ matrix with rows $\mathbf{y}_1^T, \mathbf{y}_2^T, \ldots, \mathbf{y}_m^T, 0, \ldots, 0$. Then $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{y}_i \cdot \mathbf{x} = 0$ for each $i = 1, 2, \ldots, m$; if and only if $\mathbf{x}$ is in $U^{\perp\perp} = U$.

8.1.17 d. $E^T = A^T[(AA^T)^{-}1]^T(A^T)^T = A^T[(AA^T)^T]^{-1}A = A^T[AA^T]^{-1}A = E$
$E^2 = A^T(AA^T)^{-1}AA^T(AA^T)^{-1}A = A^T(AA^T)^{-1}A = E$

Section 8.2

8.2.1 b. $\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

d. $\frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

f. $\begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

h. $\frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ -6 & 3 & 2 \end{bmatrix}$

8.2.2 We have $P^T = P^{-1}$; this matrix is lower triangular (left side) and also upper triangular (right side–see Lemma 2.7.1), and so is diagonal. But then $P = P^T = P^{-1}$, so $P^2 = I$. This implies that the diagonal entries of P are all ± 1.

8.2.5 b. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

d. $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

f. $\frac{1}{3\sqrt{2}}\begin{bmatrix} 2\sqrt{2} & 3 & 1 \\ \sqrt{2} & 0 & -4 \\ 2\sqrt{2} & -3 & 1 \end{bmatrix}$ or $\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

h. $\frac{1}{2}\begin{bmatrix} 1 & -1 & \sqrt{2} & 0 \\ -1 & 1 & \sqrt{2} & 0 \\ -1 & -1 & 0 & \sqrt{2} \\ 1 & 1 & 0 & \sqrt{2} \end{bmatrix}$

8.2.6 $P = \frac{1}{\sqrt{2}k}\begin{bmatrix} c\sqrt{2} & a & a \\ 0 & k & -k \\ -a\sqrt{2} & c & c \end{bmatrix}$

8.2.10 b. $y_1 = \frac{1}{\sqrt{5}}(-x_1 + 2x_2)$ and $y_2 = \frac{1}{\sqrt{5}}(2x_1 + x_2)$; $q = -3y_1^2 + 2y_2^2$.

8.2.11 c. $\Rightarrow$ a. By Theorem 8.2.1 let $P^{-1}AP = D = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$ where the λ_i are the eigenvalues of A. By c. we have $\lambda_i = \pm 1$ for each i, whence $D^2 = I$. But then $A^2 = (PDP^{-1})^2 = PD^2P^{-1} = I$. Since A is symmetric this is $AA^T = I$, proving a.

8.2.13 b. If $B = P^TAP = P^{-1}$, then $B^2 = P^TAPP^TAP = P^TA^2P$.

8.2.15 If $\mathbf{x}$ and $\mathbf{y}$ are respectively columns i and j of I_n, then $\mathbf{x}^T A^T \mathbf{y} = \mathbf{x}^T A \mathbf{y}$ shows that the (i, j)-entries of A^T and A are equal.

8.2.18 b. $\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 1$ and $\det \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = -1$

[*Remark*: These are the *only* 2×2 examples.]

d. Use the fact that $P^{-1} = P^T$ to show that $P^T(I-P) = -(I-P)^T$. Now take determinants and use the hypothesis that $\det P \neq (-1)^n$.

8.2.21 We have $AA^T = D$, where D is diagonal with main diagonal entries $\|R_1\|^2, \dots, \|R_n\|^2$. Hence $A^{-1} = A^T D^{-1}$, and the result follows because D^{-1} has diagonal entries $1/\|R_1\|^2, \dots, 1/\|R_n\|^2$.

8.2.23 b. Because $I-A$ and $I+A$ commute, $PP^T = (I-A)(I+A)^{-1}[(I+A)^{-1}]^T(I-A)^T = (I-A)(I+A)^{-1}(I-A)^{-1}(I+A) = I$.

Section 8.3

8.3.1 b. $U = \frac{\sqrt{2}}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

d. $U = \frac{1}{30}\begin{bmatrix} 60\sqrt{5} & 12\sqrt{5} & 15\sqrt{5} \\ 0 & 6\sqrt{30} & 10\sqrt{30} \\ 0 & 0 & 5\sqrt{15} \end{bmatrix}$

8.3.2 b. If $\lambda^k > 0$, k odd, then $\lambda > 0$.

8.3.4 If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^T A \mathbf{x} > 0$ and $\mathbf{x}^T B \mathbf{x} > 0$. Hence $\mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^T A \mathbf{x} + \mathbf{x}^T B \mathbf{x} > 0$ and $\mathbf{x}^T(rA)\mathbf{x} = r(\mathbf{x}^T A \mathbf{x}) > 0$, as $r > 0$.

8.3.6 Let $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$. Then $\mathbf{x}^T(U^T A U)\mathbf{x} = (U\mathbf{x})^T A (U\mathbf{x}) > 0$ provided $U\mathbf{x} \neq 0$. But if $U = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$, then $U\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n \neq \mathbf{0}$ because $\mathbf{x} \neq \mathbf{0}$ and the $\mathbf{c}_i$ are independent.

8.3.10 Let $P^T A P = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ where $P^T = P$. Since A is positive definite, each eigenvalue $\lambda_i > 0$. If $B = \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$ then $B^2 = D$, so $A = PB^2P^T = (PBP^T)^2$. Take $C = PBP^T$. Since C has eigenvalues $\sqrt{\lambda_i} > 0$, it is positive definite.

8.3.12 b. If A is positive definite, use Theorem 8.3.1 to write $A = U^T U$ where U is upper triangular with positive diagonal D. Then $A = (D^{-1}U)^T D^2 (D^{-1}U)$ so $A = L_1 D_1 U_1$ is such a factorization if $U_1 = D^{-1}U$, $D_1 = D^2$, and $L_1 = U_1^T$. Conversely, let $A^T = A = LDU$ be such a factorization. Then $U^T D^T L^T = A^T = A = LDU$, so $L = U^T$ by **(a)**. Hence $A = LDL^T = V^T V$ where $V = LD_0$ and D_0 is diagonal with $D_0^2 = D$ (the matrix D_0 exists because D has positive diagonal entries). Hence A is symmetric, and it is positive definite by Example 8.3.1.

Section 8.4

8.4.1 b. $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $R = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$

d. $Q = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$,

$R = \frac{1}{\sqrt{3}}\begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

8.4.2 If A has a QR-factorization, use (a). For the converse use Theorem 8.4.1.

Section 8.5

8.5.1 b. Eigenvalues $4, -1$; eigenvectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$; $\mathbf{x}_4 = \begin{bmatrix} 409 \\ -203 \end{bmatrix}$; $r_3 = 3.94$

d. Eigenvalues $\lambda_1 = \frac{1}{2}(3+\sqrt{13})$, $\lambda_2 = \frac{1}{2}(3-\sqrt{13})$; eigenvectors $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$; $\mathbf{x}_4 = \begin{bmatrix} 142 \\ 43 \end{bmatrix}$; $r_3 = 3.3027750$ (The true value is $\lambda_1 = 3.3027756$, to seven decimal places.)

8.5.2 b. Eigenvalues $\lambda_1 = \frac{1}{2}(3+\sqrt{13}) = 3.302776$, $\lambda_2 = \frac{1}{2}(3-\sqrt{13}) = -0.302776$

$A_1 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$, $Q_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$,

$R_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 3 \\ 0 & -1 \end{bmatrix}$

$A_2 = \frac{1}{10}\begin{bmatrix} 33 & -1 \\ -1 & -3 \end{bmatrix}$,

$Q_2 = \frac{1}{\sqrt{1090}}\begin{bmatrix} 33 & 1 \\ -1 & 33 \end{bmatrix}$,

$R_2 = \frac{1}{\sqrt{1090}}\begin{bmatrix} 109 & -3 \\ 0 & -10 \end{bmatrix}$

$A_3 = \frac{1}{109}\begin{bmatrix} 360 & 1 \\ 1 & -33 \end{bmatrix}$

$= \begin{bmatrix} 3.302775 & 0.009174 \\ 0.009174 & -0.302775 \end{bmatrix}$

8.5.4 Use induction on k. If $k=1$, $A_1=A$. In general $A_{k+1}=Q_k^{-1}A_kQ_k=Q_k^TA_kQ_k$, so the fact that $A_k^T=A_k$ implies $A_{k+1}^T=A_{k+1}$. The eigenvalues of A are all real (Theorem 5.5.5), so the A_k converge to an upper triangular matrix T. But T must also be symmetric (it is the limit of symmetric matrices), so it is diagonal.

Section 8.6

8.6.4 b. $t\sigma_1, \dots, t\sigma_r$.

8.6.7 If $A=U\Sigma V^T$ then Σ is invertible, so $A^{-1}=V\Sigma^{-1}U^T$ is a SVD.

8.6.8 b. First $A^TA=I_n$ so $\Sigma_A=I_n$.

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & 1 \\ 1 & -1 \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array}\right]\frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & 1 \\ -1 & 1 \end{array}\right] \\ &= \frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & -1 \\ 1 & 1 \end{array}\right]\frac{1}{\sqrt{2}}\left[\begin{array}{rr} -1 & 1 \\ 1 & 1 \end{array}\right] \\ &= \left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right] \end{aligned}$$

8.6.9 b.
$A=F$

$$=\frac{1}{5}\left[\begin{array}{rr} 3 & 4 \\ 4 & -3 \end{array}\right]\left[\begin{array}{rrrr} 20 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{array}\right]\frac{1}{2}\left[\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

8.6.13 b. If $\mathbf{x}\in\mathbb{R}^n$ then $\mathbf{x}^T(G+H)\mathbf{x}=\mathbf{x}^TG\mathbf{x}+\mathbf{x}^TH\mathbf{x}\geq 0+0=0$.

8.6.17 b. $\left[\begin{array}{rrr} \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} \end{array}\right]$

Section 8.7

8.7.1 b. $\sqrt{6}$

d. $\sqrt{13}$

8.7.2 b. Not orthogonal

d. Orthogonal

8.7.3 b. Not a subspace. For example, $i(0,\ 0,\ 1)=(0,\ 0,\ i)$ is not in U.

d. This is a subspace.

8.7.4 b. Basis $\{(i,\ 0,\ 2),\ (1,\ 0,\ -1)\}$; dimension 2

d. Basis $\{(1,\ 0,\ -2i),\ (0,\ 1,\ 1-i)\}$; dimension 2

8.7.5 b. Normal only

d. Hermitian (and normal), not unitary

f. None

h. Unitary (and normal); hermitian if and only if z is real

8.7.8 b. $U=\frac{1}{\sqrt{14}}\left[\begin{array}{cc} -2 & 3-i \\ 3+i & 2 \end{array}\right]$, $U^HAU=\left[\begin{array}{rr} -1 & 0 \\ 0 & 6 \end{array}\right]$

d. $U=\frac{1}{\sqrt{3}}\left[\begin{array}{cc} 1+i & 1 \\ -1 & 1-i \end{array}\right]$, $U^HAU=\left[\begin{array}{rr} 1 & 0 \\ 0 & 4 \end{array}\right]$

f. $U=\frac{1}{\sqrt{3}}\left[\begin{array}{ccc} \sqrt{3} & 0 & 0 \\ 0 & 1+i & 1 \\ 0 & -1 & 1-i \end{array}\right]$, $U^HAU=\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{array}\right]$

8.7.10 b. $\|\lambda Z\|^2=\langle\lambda Z,\ \lambda Z\rangle=\lambda\overline{\lambda}\langle Z,\ Z\rangle=|\lambda|^2\|Z\|^2$

8.7.11 b. If the $(k,\ k)$-entry of A is a_{kk}, then the $(k,\ k)$-entry of $\overline{A}$ is $\overline{a}_{kk}$ so the $(k,\ k)$-entry of $(\overline{A})^T=A^H$ is $\overline{a}_{kk}$. This equals a, so a_{kk} is real.

8.7.14 b. Show that $(B^2)^H=B^HB^H=(-B)(-B)=B^2$; $(iB)^H=\overline{i}B^H=(-i)(-B)=iB$.

d. If $Z=A+B$, as given, first show that $Z^H=A-B$, and hence that $A=\frac{1}{2}(Z+Z^H)$ and $B=\frac{1}{2}(Z-Z^H)$.

8.7.16 b. If U is unitary, $(U^{-1})^{-1}=(U^H)^{-1}=(U^{-1})^H$, so U^{-1} is unitary.

8.7.18 b. $H=\left[\begin{array}{rr} 1 & i \\ -i & 0 \end{array}\right]$ is hermitian but $iH=\left[\begin{array}{rr} i & -1 \\ 1 & 0 \end{array}\right]$ is not.

8.7.21 b. Let $U=\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ be real and invertible, and assume that $U^{-1}AU=\left[\begin{array}{cc} \lambda & \mu \\ 0 & v \end{array}\right]$. Then $AU=U\left[\begin{array}{cc} \lambda & \mu \\ 0 & v \end{array}\right]$, and first column entries are $c=a\lambda$ and $-a=c\lambda$. Hence λ is real (c and a are both real and are not both 0), and $(1+\lambda^2)a=0$. Thus $a=0$, $c=a\lambda=0$, a contradiction.

Section 8.8

8.8.1 b. $1^{-1}=1, 9^{-1}=9, 3^{-1}=7, 7^{-1}=3$.

d. $2^1=2, 2^2=4, 2^3=8, 2^4=16=6, 2^5=12=2, 2^6=2^2\ldots$ so $a=2^k$ if and only if $a=2, 4, 6, 8$.

8.8.2 b. If $2a=0$ in $\mathbb{Z}_{10}$, then $2a=10k$ for some integer k. Thus $a=5k$.

8.8.3 b. $11^{-1}=7$ in $\mathbb{Z}_{19}$.

8.8.6 b. $\det A=15-24=1+4=5\neq 0$ in $\mathbb{Z}_7$, so A^{-1} exists. Since $5^{-1}=3$ in $\mathbb{Z}_7$, we have
$A^{-1}=3\begin{bmatrix}3 & -6\\ 3 & 5\end{bmatrix}=3\begin{bmatrix}3 & 1\\ 3 & 5\end{bmatrix}=\begin{bmatrix}2 & 3\\ 2 & 1\end{bmatrix}$.

8.8.7 b. We have $5\cdot 3=1$ in $\mathbb{Z}_7$ so the reduction of the augmented matrix is:

$$\begin{bmatrix}3 & 1 & 4 & 3\\ 4 & 3 & 1 & 1\end{bmatrix}\to\begin{bmatrix}1 & 5 & 6 & 1\\ 4 & 3 & 1 & 1\end{bmatrix}\to\begin{bmatrix}1 & 5 & 6 & 1\\ 0 & 4 & 5 & 4\end{bmatrix}\to\begin{bmatrix}1 & 5 & 6 & 1\\ 0 & 1 & 3 & 1\end{bmatrix}\to\begin{bmatrix}1 & 0 & 5 & 3\\ 0 & 1 & 3 & 1\end{bmatrix}.$$

Hence $x=3+2t$, $y=1+4t$, $z=t$; t in $\mathbb{Z}_7$.

8.8.9 b. $(1+t)^{-1}=2+t$.

8.8.10 b. The minimum weight of C is 5, so it detects 4 errors and corrects 2 errors.

8.8.11 b. {00000, 01110, 10011, 11101}.

8.8.12 b. The code is {0000000000, 1001111000, 0101100110, 0011010111, 1100011110, 1010101111, 0110110001, 1111001001}. This has minimum distance 5 and so corrects 2 errors.

8.8.13 b. {00000, 10110, 01101, 11011} is a (5, 2)-code of minimal weight 3, so it corrects single errors.

8.8.14 b. $G=\begin{bmatrix}1 & \mathbf{u}\end{bmatrix}$ where $\mathbf{u}$ is any nonzero vector in the code. $H=\begin{bmatrix}\mathbf{u}\\ I_{n-1}\end{bmatrix}$.

Section 8.9

8.9.1 b. $A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$

d. $A=\begin{bmatrix}1 & 3 & 2\\ 3 & 1 & -1\\ 2 & -1 & 3\end{bmatrix}$

8.9.2 b. $P=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$;
$\mathbf{y}=\frac{1}{\sqrt{2}}\begin{bmatrix}x_1+x_2\\ x_1-x_2\end{bmatrix}$;
$q=3y_1^2-y_2^2$; 1, 2

d. $P=\frac{1}{3}\begin{bmatrix}2 & 2 & -1\\ 2 & -1 & 2\\ -1 & 2 & 2\end{bmatrix}$;
$\mathbf{y}=\frac{1}{3}\begin{bmatrix}2x_1+2x_2-x_3\\ 2x_1-x_2+2x_3\\ -x_1+2x_2+2x_3\end{bmatrix}$;
$q=9y_1^2+9y_2^2-9y_3^2$; 2, 3

f. $P=\frac{1}{3}\begin{bmatrix}-2 & 1 & 2\\ 2 & 2 & 1\\ 1 & -2 & 2\end{bmatrix}$;
$\mathbf{y}=\frac{1}{3}\begin{bmatrix}-2x_1+2x_2+x_3\\ x_1+2x_2-2x_3\\ 2x_1+x_2+2x_3\end{bmatrix}$;
$q=9y_1^2+9y_2^2$; 2, 2

h. $P=\frac{1}{\sqrt{6}}\begin{bmatrix}-\sqrt{2} & \sqrt{3} & 1\\ \sqrt{2} & 0 & 2\\ \sqrt{2} & \sqrt{3} & -1\end{bmatrix}$;
$\mathbf{y}=\frac{1}{\sqrt{6}}\begin{bmatrix}-\sqrt{2}x_1+\sqrt{2}x_2+\sqrt{2}x_3\\ \sqrt{3}x_1+\sqrt{3}x_3\\ x_1+2x_2-x_3\end{bmatrix}$;
$q=2y_1^2+y_2^2-y_3^2$; 2, 3

8.9.3 b. $x_1=\frac{1}{\sqrt{5}}(2x-y)$, $y_1=\frac{1}{\sqrt{5}}(x+2y)$; $4x_1^2-y_1^2=2$; hyperbola

d. $x_1=\frac{1}{\sqrt{5}}(x+2y)$, $y_1=\frac{1}{\sqrt{5}}(2x-y)$; $6x_1^2+y_1^2=1$; ellipse

8.9.4 b. Basis $\{(i, 0, i), (1, 0, -1)\}$, dimension 2

d. Basis $\{(1, 0, -2i), (0, 1, 1-i)\}$, dimension 2

8.9.7 b. $3y_1^2+5y_2^2-y_3^2-3\sqrt{2}y_1+\frac{11}{3}\sqrt{3}y_2+\frac{2}{3}\sqrt{6}y_3=7$
$y_1=\frac{1}{\sqrt{2}}(x_2+x_3)$, $y_2=\frac{1}{\sqrt{3}}(x_1+x_2-x_3)$,
$y_3=\frac{1}{\sqrt{6}}(2x_1-x_2+x_3)$

8.9.9 b. By Theorem 8.3.3 let $A=U^TU$ where U is upper triangular with positive diagonal entries. Then $q=\mathbf{x}^T(U^TU)\mathbf{x}=(U\mathbf{x})^TU\mathbf{x}=\|U\mathbf{x}\|^2$.

Section 9.1

9.1.1 b. $\begin{bmatrix}a\\ 2b-c\\ c-b\end{bmatrix}$

d. $\frac{1}{2}\begin{bmatrix} a-b \\ a+b \\ -a+3b+2c \end{bmatrix}$

9.1.2 b. Let $\mathbf{v} = a+bx+cx^2$. Then $C_D[T(\mathbf{v})] = M_{DB}(T)C_B(\mathbf{v}) =$

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+b+3c \\ -a-2c \end{bmatrix}$$

Hence

$$\begin{aligned} T(\mathbf{v}) &= (2a+b+3c)(1,\, 1)+(-a-2c)(0,\, 1) \\ &= (2a+b+3c,\, a+b+c). \end{aligned}$$

9.1.3 b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

9.1.4 b. $\begin{bmatrix} 1 & 2 \\ 5 & 3 \\ 4 & 0 \\ 1 & 1 \end{bmatrix}$;

$$C_D[T(a,\, b)] = \begin{bmatrix} 1 & 2 \\ 5 & 3 \\ 4 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} b \\ a-b \end{bmatrix} = \begin{bmatrix} 2a-b \\ 3a+2b \\ 4b \\ a \end{bmatrix}$$

d. $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$; $C_D[T(a+bx+cx^2)] =$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a+b-c \\ a+b+c \end{bmatrix}$$

f. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $C_D\left(T\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b+c \\ b+c \\ d \end{bmatrix}$$

9.1.5 b. $M_{ED}(S)M_{DB}(T) =$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix} = M_{EB}(ST)$$

d. $M_{ED}(S)M_{DB}(T) =$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = M_{EB}(ST)$$

9.1.7 b.

$T^{-1}(a,\, b,\, c) = \frac{1}{2}(b+c-a,\, a+c-b,\, a+b-c)$;

$$M_{DB}(T) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix};$$

$$M_{BD}(T^{-1}) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

d. $T^{-1}(a,\, b,\, c) = (a-b)+(b-c)x+cx^2$;

$$M_{DB}(T) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix};$$

$$M_{BD}(T^{-1}) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

9.1.8 b. $M_{DB}(T^{-1}) = [M_{BD}(T)]^{-1} =$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence $C_B[T^{-1}(a,\, b,\, c,\, d)] = M_{BD}(T^{-1})C_D(a,\, b,\, c,\, d) =$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ b-c \\ c \\ d \end{bmatrix}, \text{ so}$$

$$T^{-1}(a,\, b,\, c,\, d) = \begin{bmatrix} a-b & b-c \\ c & d \end{bmatrix}.$$

9.1.12 Have $C_D[T(\mathbf{e}_j)]$ = column j of I_n. Hence $M_{DB}(T) = \begin{bmatrix} C_D[T(\mathbf{e}_1)] & C_D[T(\mathbf{e}_2)] & \cdots & C_D[T(\mathbf{e}_n)] \end{bmatrix} = I_n$.

9.1.16 b. If D is the standard basis of $\mathbb{R}^{n+1}$ and $B = \{1,\, x,\, x^2,\, \dots,\, x^n\}$, then $M_{DB}(T) = \begin{bmatrix} C_D[T(1)] & C_D[T(x)] & \cdots & C_D[T(x^n)] \end{bmatrix} =$

$$\begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{bmatrix}.$$

This matrix has nonzero determinant by Theorem 3.2.7 (since the a_i are distinct), so T is an isomorphism.

9.1.20 d. $[(S+T)R](\mathbf{v}) = (S+T)(R(\mathbf{v})) = S[(R(\mathbf{v}))] + T[(R(\mathbf{v}))] = SR(\mathbf{v}) + TR(\mathbf{v}) = [SR+TR](\mathbf{v})$ holds for all $\mathbf{v}$ in V. Hence $(S+T)R = SR+TR$.

9.1.21 b. If $\mathbf{w}$ lies in $\text{im}\,(S+T)$, then $\mathbf{w} = (S+T)(\mathbf{v})$ for some $\mathbf{v}$ in V. But then $\mathbf{w} = S(\mathbf{v}) + T(\mathbf{v})$, so $\mathbf{w}$ lies in $\text{im}\,S + \text{im}\,T$.

9.1.22 b. If $X \subseteq X_1$, let T lie in X_1^0. Then $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in X_1, whence $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in X. Thus T is in X^0 and we have shown that $X_1^0 \subseteq X^0$.

9.1.24 b. R is linear means $S_{\mathbf{v}+\mathbf{w}} = S_\mathbf{v} + S_\mathbf{w}$ and $S_{a\mathbf{v}} = aS_\mathbf{v}$. These are proved as follows: $S_{\mathbf{v}+\mathbf{w}}(r) = r(\mathbf{v}+\mathbf{w}) = r\mathbf{v} + r\mathbf{w} = S\mathbf{v}(r) + S\mathbf{w}(r) = (S\mathbf{v} + S\mathbf{w})(r)$, and $S_{a\mathbf{v}}(r) = r(a\mathbf{v}) = a(r\mathbf{v}) = (aS_\mathbf{v})(r)$ for all r in $\mathbb{R}$. To show R is one-to-one, let $R(\mathbf{v}) = \mathbf{0}$. This means $S_\mathbf{v} = 0$ so $0 = S_\mathbf{v}(r) = r\mathbf{v}$ for all r. Hence $\mathbf{v} = \mathbf{0}$ (take $r = 1$). Finally, to show R is onto, let T lie in $\mathbf{L}(\mathbb{R}, V)$. We must find $\mathbf{v}$ such that $R(\mathbf{v}) = T$, that is $S_\mathbf{v} = T$. In fact, $\mathbf{v} = T(1)$ works since then $T(r) = T(r \cdot 1) = rT(1) = r\mathbf{v} = S_\mathbf{v}(r)$ holds for all r, so $T = S_\mathbf{v}$.

9.1.25 b. Given $T : \mathbb{R} \to V$, let $T(1) = a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n$, a_i in $\mathbb{R}$. For all r in $\mathbb{R}$, we have $(a_1S_1 + \cdots + a_nS_n)(r) = a_1S_1(r) + \cdots + a_nS_n(r) = (a_1r\mathbf{b}_1 + \cdots + a_nr\mathbf{b}_n) = rT(1) = T(r)$. This shows that $a_1S_1 + \cdots + a_nS_n = T$.

9.1.27 b. Write $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$, v_j in $\mathbb{R}$. Apply E_i to get $E_i(\mathbf{v}) = v_1E_i(\mathbf{b}_1) + \cdots + v_nE_i(\mathbf{b}_n) = v_i$ by the definition of the E_i.

Section 9.2

9.2.1 b. $\frac{1}{2}\begin{bmatrix} -3 & -2 & 1 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

9.2.4 b. $P_{B\leftarrow D} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,
$P_{D\leftarrow B} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ -1 & -1 & 2 \end{bmatrix}$,
$P_{E\leftarrow D} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$, $P_{E\leftarrow B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

9.2.5 b. $A = P_{D\leftarrow B}$, where $B = \{(1, 2, -1), (2, 3, 0), (1, 0, 2)\}$. Hence $A^{-1} = P_{B\leftarrow D} = \begin{bmatrix} 6 & -4 & -3 \\ -4 & 3 & 2 \\ 3 & -2 & -1 \end{bmatrix}$

9.2.7 b. $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

9.2.8 b. $B = \left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

9.2.9 b. $c_T(x) = x^2 - 6x - 1$

d. $c_T(x) = x^3 + x^2 - 8x - 3$

f. $c_T(x) = x^4$

9.2.12 Define $T_A : \mathbb{R}^n \to \mathbb{R}^n$ by $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. If $\text{null}\,A = \text{null}\,B$, then $\ker(T_A) = \text{null}\,A = \text{null}\,B = \ker(T_B)$ so, by Exercise 7.3.28, $T_A = ST_B$ for some isomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$. If B_0 is the standard basis of $\mathbb{R}^n$, we have $A = M_{B_0}(T_A) = M_{B_0}(ST_B) = M_{B_0}(S)M_{B_0}(T_B) = UB$ where $U = M_{B_0}(S)$ is invertible by Theorem 9.2.1. Conversely, if $A = UB$ with U invertible, then $A\mathbf{x} = \mathbf{0}$ if and only $B\mathbf{x} = \mathbf{0}$, so $\text{null}\,A = \text{null}\,B$.

9.2.16 b. Showing $S(w+v) = S(w) + S(v)$ means $M_B(T_{w+v}) = M_B(T_w) + M_B(T_v)$. If $B = \{b_1, b_2\}$, then column j of $M_B(T_{w+v})$ is $C_B[(w+v)b_j] = C_B(wb_j + vb_j) = C_B(wb_j) + C_B(vb_j)$ because C_B is linear. This is column j of $M_B(T_w) + M_B(T_v)$. Similarly $M_B(T_{aw}) = aM_B(T_w)$; so $S(aw) = aS(w)$. Finally $T_wT_v = T_{wv}$ so $S(wv) = M_B(T_wT_v) = M_B(T_w)M_B(T_v) = S(w)S(v)$ by Theorem 9.2.1.

Section 9.3

9.3.2 b. $T(U) \subseteq U$, so $T[T(U)] \subseteq T(U)$.

9.3.3 b. If $\mathbf{v}$ is in $S(U)$, write $\mathbf{v} = S(\mathbf{u})$, $\mathbf{u}$ in U. Then $T(\mathbf{v}) = T[S(\mathbf{u})] = (TS)(\mathbf{u}) = (ST)(\mathbf{u}) = S[T(\mathbf{u})]$ and this lies in $S(U)$ because $T(\mathbf{u})$ lies in U (U is T-invariant).

9.3.6 Suppose U is T-invariant for every T. If $U \neq 0$, choose $\mathbf{u} \neq \mathbf{0}$ in U. Choose a basis $B = \{\mathbf{u}, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ of V containing $\mathbf{u}$. Given any $\mathbf{v}$ in V, there is (by Theorem 7.1.3) a linear transformation $T : V \to V$ such that $T(\mathbf{u}) = \mathbf{v}$, $T(\mathbf{u}_2) = \cdots = T(\mathbf{u}_n) = \mathbf{0}$. Then $\mathbf{v} = T(\mathbf{u})$ lies in U because U is T-invariant. This shows that $V = U$.

9.3.8 b. $T(1-2x^2) = 3+3x-3x^2 = 3(1-2x^2)+3(x+x^2)$ and $T(x+x^2) = -(1-2x^2)$, so both are in U. Hence U is T-invariant by Example 9.3.3. If $B = \{1-2x^2,\ x+x^2,\ x^2\}$ then $M_B(T) = \begin{bmatrix} 3 & -1 & 1 \\ 3 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, so $c_T(x) = \det \begin{bmatrix} x-3 & 1 & -1 \\ -3 & x & -1 \\ 0 & 0 & x-3 \end{bmatrix} = (x-3)\det \begin{bmatrix} x-3 & 1 \\ -3 & x \end{bmatrix} = (x-3)(x^2-3x+3)$

9.3.9 b. Suppose $\mathbb{R}\mathbf{u}$ is T_A-invariant where $\mathbf{u} \neq \mathbf{0}$. Then $T_A(\mathbf{u}) = r\mathbf{u}$ for some r in $\mathbb{R}$, so $(rI-A)\mathbf{u} = \mathbf{0}$. But $\det(rI-A) = (r-\cos\theta)^2 + sin^2\theta \neq 0$ because $0 < \theta < \pi$. Hence $\mathbf{u} = \mathbf{0}$, a contradiction.

9.3.10 b. $U = \text{span}\{(1, 1, 0, 0), (0, 0, 1, 1)\}$ and $W = \text{span}\{(1, 0, 1, 0), (0, 1, 0, -1)\}$, and these four vectors form a basis of $\mathbb{R}^4$. Use Example 9.3.9.

d. $U = \text{span}\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$ and $W = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$ and these vectors are a basis of $\mathbf{M}_{22}$. Use Example 9.3.9.

9.3.14 The fact that U and W are subspaces is easily verified using the subspace test. If A lies in $U \cap V$, then $A = AE = 0$; that is, $U \cap V = 0$. To show that $\mathbf{M}_{22} = U + V$, choose any A in $\mathbf{M}_{22}$. Then $A = AE + (A - AE)$, and AE lies in U [because $(AE)E = AE^2 = AE$], and $A - AE$ lies in W [because $(A-AE)E = AE - AE^2 = 0$].

9.3.17 b. By **(a)** it remains to show $U + W = V$; we show that $\dim(U+W) = n$ and invoke Theorem 6.4.2. But $U + W = U \oplus W$ because $U \cap W = 0$, so $\dim(U+W) = \dim U + \dim W = n$.

9.3.18 b. First, $\ker(T_A)$ is T_A-invariant. Let $U = \mathbb{R}\mathbf{p}$ be T_A-invariant. Then $T_A(\mathbf{p})$ is in U, say $T_A(\mathbf{p}) = \lambda\mathbf{p}$. Hence $A\mathbf{p} = \lambda\mathbf{p}$ so λ is an eigenvalue of A. This means that $\lambda = 0$ by **(a)**, so $\mathbf{p}$ is in $\ker(T_A)$. Thus $U \subseteq \ker(T_A)$. But $\dim[\ker(T_A)] \neq 2$ because $T_A \neq 0$, so $\dim[\ker(T_A)] = 1 = \dim(U)$. Hence $U = \ker(T_A)$.

9.3.20 Let B_1 be a basis of U and extend it to a basis B of V. Then $M_B(T) = \begin{bmatrix} M_{B_1}(T) & Y \\ 0 & Z \end{bmatrix}$, so $c_T(x) = \det[xI - M_B(T)] = \det[xI - M_{B_1}(T)]\det[xI - Z] = c_{T1}(x)q(x)$.

9.3.22 b. $T^2[p(x)] = p[-(-x)] = p(x)$, so $T^2 = 1$; $B = \{1, x^2;\ x, x^3\}$

d. $T^2(a, b, c) = T(-a+2b+c, b+c, -c) = (a, b, c)$, so $T^2 = 1$; $B = \{(1, 1, 0);\ (1, 0, 0), (0, -1, 2)\}$

9.3.23 b. Use the Hint and Exercise 9.3.2.

9.3.25 b. $T^2(a, b, c) = T(a+2b, 0, 4b+c) = (a+2b, 0, 4b+c) = T(a, b, c)$, so $T^2 = T$; $B = \{(1, 0, 0), (0, 0, 1);\ (2, -1, 4)\}$

9.3.29 b. $T_{f,\,\mathbf{z}}[T_{f,\,\mathbf{z}}(\mathbf{v})] = T_{f,\,\mathbf{z}}[f(\mathbf{v})\mathbf{z}] = f[f(\mathbf{v})\mathbf{z}]\mathbf{z} = f(\mathbf{v})\{f[\mathbf{z}]\mathbf{z}\} = f(\mathbf{v})f(\mathbf{z})\mathbf{z}$. This equals $T_{f,\,\mathbf{z}}(\mathbf{v}) = f(\mathbf{v})\mathbf{z}$ for all $\mathbf{v}$ if and only if $f(\mathbf{v})f(\mathbf{z}) = f(\mathbf{v})$ for all $\mathbf{v}$. Since $f \neq 0$, this holds if and only if $f(z) = 1$.

9.3.30 b. If $A = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$ where $U\mathbf{p}_i = \lambda\mathbf{p}_i$ for each i, then $UA = \lambda A$. Conversely, $UA = \lambda A$ means that $U\mathbf{p} = \lambda\mathbf{p}$ for every column $\mathbf{p}$ of A.

Section 10.1

10.1.1 b. P5 fails.

d. P5 fails.

f. P5 fails.

10.1.2 Axioms P1–P5 hold in U because they hold in V.

10.1.3 b. $\frac{1}{\sqrt{\pi}}f$

d. $\frac{1}{\sqrt{17}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

10.1.4 b. $\sqrt{3}$

d. $\sqrt{3\pi}$

10.1.8 P1 and P2 are clear since $f(i)$ and $g(i)$ are real numbers.

$$\begin{aligned} \text{P3: } \langle f+g,\ h\rangle &= \sum_i (f+g)(i)\cdot h(i) \\ &= \sum_i (f(i)+g(i))\cdot h(i) \\ &= \sum_i [f(i)h(i) + g(i)h(i)] \\ &= \sum_i f(i)h(i) + \sum_i g(i)h(i) \\ &= \langle f,\ h\rangle + \langle g,\ h\rangle. \end{aligned}$$

$$\text{P4: } \quad \langle rf,\ g\rangle = \sum_i (rf)(i)\cdot g(i)$$

$$= \sum_i rf(i) \cdot g(i)$$
$$= r\sum_i f(i) \cdot g(i)$$
$$= r\langle f,\ g\rangle$$

P5: If $f \neq 0$, then $\langle f,\ f\rangle = \sum_i f(i)^2 > 0$ because some $f(i) \neq 0$.

10.1.12 b. $\langle \mathbf{v},\ \mathbf{v}\rangle = 5v_1^2 - 6v_1v_2 + 2v_2^2 = \frac{1}{5}[(5v_1 - 3v_2)^2 + v_2^2]$

d. $\langle \mathbf{v},\ \mathbf{v}\rangle = 3v_1^2 + 8v_1v_2 + 6v_2^2 = \frac{1}{3}[(3v_1 + 4v_2)^2 + 2v_2^2]$

10.1.13 b. $\left[\begin{array}{rr} 1 & -2 \\ -2 & 1 \end{array}\right]$

d. $\left[\begin{array}{rrr} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 5 \end{array}\right]$

10.1.14 By the condition, $\langle \mathbf{x},\ \mathbf{y}\rangle = \frac{1}{2}\langle \mathbf{x}+\mathbf{y},\ \mathbf{x}+\mathbf{y}\rangle = 0$ for all $\mathbf{x}$, $\mathbf{y}$. Let $\mathbf{e}_i$ denote column i of I. If $A = [a_{ij}]$, then $a_{ij} = \mathbf{e}_i^T A \mathbf{e}_j = \{\mathbf{e}_i,\ \mathbf{e}_j\} = 0$ for all i and j.

10.1.16 b. -15

10.1.20 **1.** Using P2: $\langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{v}+\mathbf{w},\ \mathbf{u}\rangle = \langle \mathbf{v},\ \mathbf{u}\rangle + \langle \mathbf{w},\ \mathbf{u}\rangle = \langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{u},\ \mathbf{w}\rangle$.

2. Using P2 and P4: $\langle \mathbf{v},\ r\mathbf{w}\rangle = \langle r\mathbf{w},\ \mathbf{v}\rangle = r\langle \mathbf{w},\ \mathbf{v}\rangle = r\langle \mathbf{v},\ \mathbf{w}\rangle$.

3. Using P3: $\langle \mathbf{0},\ \mathbf{v}\rangle = \langle \mathbf{0}+\mathbf{0},\ \mathbf{v}\rangle = \langle \mathbf{0},\ \mathbf{v}\rangle + \langle \mathbf{0},\ \mathbf{v}\rangle$, so $\langle \mathbf{0},\ \mathbf{v}\rangle = 0$. The rest is P2.

4. Assume that $\langle \mathbf{v},\ \mathbf{v}\rangle = 0$. If $\mathbf{v} \neq \mathbf{0}$ this contradicts P5, so $\mathbf{v} = \mathbf{0}$. Conversely, if $\mathbf{v} = \mathbf{0}$, then $\langle \mathbf{v},\ \mathbf{v}\rangle = 0$ by Part 3 of this theorem.

10.1.22 b. $15\|\mathbf{u}\|^2 - 17\langle \mathbf{u},\ \mathbf{v}\rangle - 4\|\mathbf{v}\|^2$

d. $\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v},\ \mathbf{u}+\mathbf{v}\rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u},\ \mathbf{v}\rangle + \|\mathbf{v}\|^2$

10.1.26 b. $\{(1,\ 1,\ 0),\ (0,\ 2,\ 1)\}$

10.1.28 $\langle \mathbf{v}-\mathbf{w},\ \mathbf{v}_i\rangle = \langle \mathbf{v},\ \mathbf{v}_i\rangle - \langle \mathbf{w},\ \mathbf{v}_i\rangle = 0$ for each i, so $\mathbf{v} = \mathbf{w}$ by Exercise 10.1.27.

10.1.29 b. If $\mathbf{u} = (\cos\theta,\ \sin\theta)$ in $\mathbb{R}^2$ (with the dot product) then $\|\mathbf{u}\| = 1$. Use **(a)** with $\mathbf{v} = (x,\ y)$.

Section 10.2

10.2.1 b. $\frac{1}{14}\left\{(6a+2b+6c)\left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right] + (7c-7a)\left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right] + (a-2b+c)\left[\begin{array}{r} 1 \\ -6 \\ 1 \end{array}\right]\right\}$

d. $\left(\frac{a+d}{2}\right)\left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array}\right] + \left(\frac{a-d}{2}\right)\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] + \left(\frac{b+c}{2}\right)\left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right] + \left(\frac{b-c}{2}\right)\left[\begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array}\right]$

10.2.2 b. $\{(1,\ 1,\ 1),\ (1,\ -5,\ 1),\ (3,\ 0,\ -2)\}$

10.2.3 b. $\left\{\left[\begin{array}{rr} 1 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{rr} 1 & -2 \\ 3 & 1 \end{array}\right], \left[\begin{array}{rr} 1 & -2 \\ -2 & 1 \end{array}\right], \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]\right\}$

10.2.4 b. $\{1,\ x-1,\ x^2 - 2x + \frac{2}{3}\}$

10.2.6 b. $U^\perp = \text{span}\{[\ 1\ \ -1\ \ 0\ \ 0\ ],\ [\ 0\ \ 0\ \ 1\ \ 0\ ],\ [\ 0\ \ 0\ \ 0\ \ 1\ ]\}$, $\dim U^\perp = 3$, $\dim U = 1$

d. $U^\perp = \text{span}\{2-3x,\ 1-2x^2\}$, $\dim U^\perp = 2$, $\dim U = 1$

f. $U^\perp = \text{span}\left\{\left[\begin{array}{rr} 1 & -1 \\ -1 & 0 \end{array}\right]\right\}$, $\dim U^\perp = 1$, $\dim U = 3$

10.2.7 b. $U = \text{span}\left\{\left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{rr} 1 & 1 \\ 1 & -1 \end{array}\right], \left[\begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array}\right]\right\}$;

$\text{proj}_U A = \left[\begin{array}{rr} 3 & 0 \\ 2 & 1 \end{array}\right]$

10.2.8 b. $U = \text{span}\{1,\ 5-3x^2\}$; $\text{proj}_U\ x = \frac{3}{13}(1+2x^2)$

10.2.9 b. $B = \{1,\ 2x-1\}$ is an orthogonal basis of U because $\int_0^1 (2x-1)dx = 0$. Using it, we get $\text{proj}_U\ (x^2+1) = x + \frac{5}{6}$, so $x^2 + 1 = (x + \frac{5}{6}) + (x^2 - x + \frac{1}{6})$.

10.2.11 b. This follows from $\langle \mathbf{v}+\mathbf{w},\ \mathbf{v}-\mathbf{w}\rangle = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2$.

10.2.14 b. $U^\perp \subseteq \{\mathbf{u}_1,\ \dots,\ \mathbf{u}_m\}^\perp$ because each $\mathbf{u}_i$ is in U. Conversely, if $\langle \mathbf{v},\ \mathbf{u}_i\rangle = 0$ for each i, and $\mathbf{u} = r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m$ is any vector in U, then $\langle \mathbf{v},\ \mathbf{u}\rangle = r_1\langle \mathbf{v},\ \mathbf{u}_1\rangle + \cdots + r_m\langle \mathbf{v},\ \mathbf{u}_m\rangle = 0$.

10.2.18 b. $\text{proj}_U\ (-5,\ 4,\ -3) = (-5,\ 4,\ -3)$; $\text{proj}_U\ (-1,\ 0,\ 2) = \frac{1}{38}(-17,\ 24,\ 73)$

10.2.19 b. The plane is $U = \{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{n} = 0\}$ so span $\left\{\mathbf{n} \times \mathbf{w},\ \mathbf{w} - \frac{\mathbf{n} \cdot \mathbf{w}}{\|\mathbf{n}\|^2}\mathbf{n}\right\} \subseteq U$. This is equality because both spaces have dimension 2 (using **(a)**).

10.2.20 b. $C_E(\mathbf{b}_i)$ is column i of P. Since $C_E(\mathbf{b}_i) \cdot C_E(\mathbf{b}_j) = \langle \mathbf{b}_i, \mathbf{b}_j \rangle$ by **(a)**, the result follows.

10.2.23 b. If $U = \text{span}\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$, then $\text{proj}_U \mathbf{v} = \sum_{i=1}^{m} \frac{\langle \mathbf{v}_1, \mathbf{f}_i \rangle}{\|\mathbf{f}_i\|^2}\mathbf{f}_i$ by Theorem 10.2.7. Hence $\|\text{proj}_U \mathbf{v}\|^2 = \sum_{i=1}^{m} \frac{\langle \mathbf{v}_1, \mathbf{f}_i \rangle}{\|\mathbf{f}_i\|^2}\mathbf{f}_i$ by Pythagoras' theorem. Now use **(a)**.

Section 10.3

10.3.1 b.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\};$$

$$M_B(T) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

10.3.4 b. $\langle \mathbf{v}, (rT)\mathbf{w} \rangle = \langle \mathbf{v}, rT(\mathbf{w}) \rangle = r\langle \mathbf{v}, T(\mathbf{w}) \rangle = r\langle T(\mathbf{v}), \mathbf{w} \rangle = \langle rT(\mathbf{v}), \mathbf{w} \rangle = \langle (rT)(\mathbf{v}), \mathbf{w} \rangle$

d. Given $\mathbf{v}$ and $\mathbf{w}$, write $T^{-1}(\mathbf{v}) = \mathbf{v}_1$ and $T^{-1}(\mathbf{w}) = \mathbf{w}_1$. Then $\langle T^{-1}(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}_1, T(\mathbf{w}_1) \rangle = \langle T(\mathbf{v}_1), \mathbf{w}_1 \rangle = \langle \mathbf{v}, T^{-1}(\mathbf{w}) \rangle$.

10.3.5 b. If $B_0 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, then $M_{B_0}(T) = \begin{bmatrix} 7 & -1 & 0 \\ -1 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has an orthonormal basis of eigenvectors $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Hence an orthonormal basis of eigenvectors of T is $\left\{ \frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{\sqrt{2}}(1, -1, 0), (0, 0, 1) \right\}$.

d. If $B_0 = \{1, x, x^2\}$, then $M_{B_0}(T) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ has an orthonormal basis of eigenvectors $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$. Hence an orthonormal basis of eigenvectors of T is $\left\{ x, \frac{1}{\sqrt{2}}(1 + x^2), \frac{1}{\sqrt{2}}(1 - x^2) \right\}$.

10.3.7 b. $M_B(T) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, so

$$c_T(x) = \det \begin{bmatrix} xI_2 - A & 0 \\ 0 & xI_2 - A \end{bmatrix} = [c_A(x)]^2.$$

10.3.12 (1) $\Rightarrow$ (2). If $B = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ is an orthonormal basis of V, then $M_B(T) = [a_{ij}]$ where $a_{ij} = \langle \mathbf{f}_i, T(\mathbf{f}_j) \rangle$ by Theorem 10.3.2. If (1) holds, then $a_{ji} = \langle \mathbf{f}_j, T(\mathbf{f}_i) \rangle = -\langle T(\mathbf{f}_j), \mathbf{f}_i \rangle = -\langle \mathbf{f}_i, T(\mathbf{f}_j) \rangle = -a_{ij}$. Hence $[M_V(T)]^T = -M_V(T)$, proving (2).

10.3.14 c. The coefficients in the definition of $T'(\mathbf{f}_j) = \sum_{i=1}^{n} \langle \mathbf{f}_j, T(\mathbf{f}_i) \rangle \mathbf{f}_i$ are the entries in the jth column $C_B[T'(\mathbf{f}_j)]$ of $M_B(T')$. Hence $M_B(T') = [\langle \mathbf{f}_j, T(\mathbf{f}_i) \rangle]$, and this is the transpose of $M_B(T)$ by Theorem 10.3.2.

Section 10.4

10.4.2 b. Rotation through π

d. Reflection in the line $y = -x$

f. Rotation through $\frac{\pi}{4}$

10.4.3 b. $c_T(x) = (x - 1)(x^2 + \frac{3}{2}x + 1)$. If $\mathbf{e} = \begin{bmatrix} 1 & \sqrt{3} & \sqrt{3} \end{bmatrix}^T$, then T is a rotation about $\mathbb{R}\mathbf{e}$.

d. $c_T(x) = (x + 1)(x + 1)^2$. Rotation (of π) about the x axis.

f. $c_T(x) = (x + 1)(x^2 - \sqrt{2}x + 1)$. Rotation (of $-\frac{\pi}{4}$) about the y axis followed by a reflection in the $x - z$ plane.

10.4.6 If $\|\mathbf{v}\| = \|(aT)(\mathbf{v})\| = |a|\|T(\mathbf{v})\| = |a|\|\mathbf{v}\|$ for some $\mathbf{v} \neq \mathbf{0}$, then $|a| = 1$ so $a = \pm 1$.

10.4.12 b. Assume that $S = S_{\mathbf{u}} \circ T$, $\mathbf{u} \in V$, T an isometry of V. Since T is onto (by Theorem 10.4.2), let $\mathbf{u} = T(\mathbf{w})$ where $\mathbf{w} \in V$. Then for any $\mathbf{v} \in V$, we have $(T \circ S_{\mathbf{w}}) = T(\mathbf{w} + \mathbf{v}) = T(\mathbf{w}) + T(\mathbf{w}) = S_{T(\mathbf{w})}(T(\mathbf{v})) = (S_{T(\mathbf{w})} \circ T)(\mathbf{v})$, and it follows that $T \circ S_{\mathbf{w}} = S_{T(\mathbf{w})} \circ T$.

Section 10.5

10.5.1 b. $\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2}\right]$

d. $\frac{\pi}{4} + \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5}\right] - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2}\right]$

10.5.2 b. $\frac{2}{\pi} - \frac{8}{\pi}\left[\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1}\right]$

10.5.4 $\int \cos kx \cos lx\, dx = \frac{1}{2}\left[\frac{\sin[(k+l)x]}{k+l} - \frac{\sin[(k-l)x]}{k-l}\right]_0^{\pi} = 0$ provided that $k \neq l$.

Section 11.1

11.1.1 b. $c_A(x) = (x+1)^3$;

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix};$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

d. $c_A(x) = (x-1)^2(x+2)$;

$$P = \begin{bmatrix} -1 & 0 & -1 \\ 4 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix};$$

$$P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

f. $c_A(x) = (x+1)^2(x-1)^2$;

$$P = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix};$$

$$P^{-1}AP = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11.1.4 If B is any ordered basis of V, write $A = M_B(T)$. Then $c_T(x) = c_A(x) = a_0 + a_1x + \cdots + a_nx^n$ for scalars a_i in $\mathbb{R}$. Since M_B is linear and $M_B(T^k) = M_B(T)^k$, we have $M_B[c_T(T)] = M_V[a_0 + a_1T + \cdots + a_nT^n] = a_0I + a_1A + \cdots + a_nA^n = c_A(A) = 0$ by the Cayley-Hamilton theorem. Hence $c_T(T) = 0$ because M_B is one-to-one.

Section 11.2

11.2.2

$$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$$

Appendix A

A.1 b. $x = 3$

d. $x = \pm 1$

A.2 b. $10 + i$

d. $\frac{11}{26} + \frac{23}{26}i$

f. $2 - 11i$

h. $8 - 6i$

A.3 b. $\frac{11}{5} + \frac{3}{5}i$

d. $\pm(2 - i)$

f. $1 + i$

A.4 b. $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

d. $2, \frac{1}{2}$

A.5 b. $-2,\ 1 \pm \sqrt{3}i$

d. $\pm 2\sqrt{2},\ \pm 2\sqrt{i}$

A.6 b. $x^2 - 4x + 13;\ 2 + 3i$

d. $x^2 - 6x + 25;\ 3 + 4i$

A.8 $x^4 - 10x^3 + 42x^2 - 82x + 65$

A.10 b. $(-2)^2 + 2i - (4 - 2i) = 0;\ 2 - i$

d. $(-2+i)^2 + 3(1-i)(-1+2i) - 5i = 0;\ -1 + 2i$

A.11 b. $-i,\ 1 + i$

d. $2 - i,\ 1 - 2i$

A.12 b. Circle, centre at 1, radius 2

d. Imaginary axis

f. Line $y = mx$

A.18 b. $4e^{-\pi i/2}$

d. $8e^{2\pi i/3}$

f. $6\sqrt{2}e^{3\pi i/4}$

A.19 b. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

d. $1 - i$

f. $\sqrt{3} - 3i$

A.20 b. $-\frac{1}{32} + \frac{\sqrt{3}}{32}i$

d. $-32i$

f. $-2^{16}(1 + i)$

A.23 b. $\pm\frac{\sqrt{2}}{2}(\sqrt{3} + i),\ \pm\frac{\sqrt{2}}{2}(-1 + \sqrt{3}i)$

d. $\pm 2i,\ \pm(\sqrt{3} + i),\ \pm(\sqrt{3} - i)$

A.26 b. The argument in (a) applies using $\beta = \frac{2\pi}{n}$. Then $1 + z + \cdots + z^{n-1} = \frac{1-z^n}{1-z} = 0$.

Appendix B

B.1 b. If $m = 2p$ and $n = 2q + 1$ where p and q are integers, then $m + n = 2(p + q) + 1$ is odd. The converse is false: $m = 1$ and $n = 2$ is a counterexample.

d. $x^2 - 5x + 6 = (x - 2)(x - 3)$ so, if this is zero, then $x = 2$ or $x = 3$. The converse is true: each of 2 and 3 satisfies $x^2 - 5x + 6 = 0$.

B.2 b. This implication is true. If $n = 2t + 1$ where t is an integer, then $n^2 = 4t^2 + 4t + 1 = 4t(t + 1) + 1$. Now t is either even or odd, say $t = 2m$ or $t = 2m + 1$. If $t = 2m$, then $n^2 = 8m(2m + 1) + 1$; if $t = 2m + 1$, then $n^2 = 8(2m + 1)(m + 1) + 1$. Either way, n^2 has the form $n^2 = 8k + 1$ for some integer k.

B.3 b. Assume that the statement "one of m and n is greater than 12" is false. Then both $n \leq 12$ and $m \leq 12$, so $n + m \leq 24$, contradicting the hypothesis that $n + m = 25$. This proves the implication. The converse is false: $n = 13$ and $m = 13$ is a counterexample.

d. Assume that the statement "m is even or n is even" is false. Then both m and n are odd, so mn is odd, contradicting the hypothesis. The converse is true: If m or n is even, then mn is even.

B.4 b. If x is irrational and y is rational, assume that $x + y$ is rational. Then $x = (x + y) - y$ is the difference of two rationals, and so is rational, contrary to the hypothesis.

B.5 b. $n = 10$ is a counterexample because $10^3 = 1000$ while $2^{10} = 1024$, so the statement $n^3 \geq 2^n$ is false if $n = 10$. Note that $n^3 \geq 2^n$ *does* hold for $2 \leq n \leq 9$.

Appendix C

C.6 $\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$

C.14
$2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}} = \frac{2\sqrt{n^2+n+1}}{\sqrt{n+1}} - 1 < \frac{2(n+1)}{\sqrt{n+1}} - 1 = 2\sqrt{n+1} - 1$

C.18 If $n^3 - n = 3k$, then
$(n + 1)^3 - (n + 1) = 3k + 3n^2 + 3n = 3(k + n^2 + n)$

C.20 $B_n = (n + 1)! - 1$

C.22 b. Verify each of $S_1, S_2, \dots, S_8$.