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Foreward

Mathematics education at the beginning university level is closely tied to the traditional publishers. In my opinion, it gives them too much control of both cost and content. The main goal of most publishers is profit, and the result has been a sales-driven business model as opposed to a pedagogical one. This results in frequent new “editions” of textbooks motivated largely to reduce the sale of used books rather than to update content quality. It also introduces copyright restrictions which stifle the creation and use of new pedagogical methods and materials. The overall result is high cost textbooks which may not meet the evolving educational needs of instructors and students.

To be fair, publishers do try to produce material that reflects new trends. But their goal is to sell books and not necessarily to create tools for student success in mathematics education. Sadly, this has led to a model where the primary choice for adapting to (or initiating) curriculum change is to find a different commercial textbook. My editor once said that the text that is adopted is often everyone’s third choice.

Of course instructors can produce their own lecture notes, and have done so for years, but this remains an onerous task. The publishing industry arose from the need to provide authors with copy-editing, editorial, and marketing services, as well as extensive reviews of prospective customers to ascertain market trends and content updates. These are necessary skills and services that the industry continues to offer.

Authors of open educational resources (OER) including (but not limited to) textbooks and lecture notes, cannot afford this on their own. But they do have two great advantages: The cost to students is significantly lower, and open licenses return content control to instructors. Through editable file formats and open licenses, OER can be developed, maintained, reviewed, edited, and improved by a variety of contributors. Instructors can now respond to curriculum change by revising and reordering material to create content that meets the needs of their students. While editorial and quality control remain daunting tasks, great strides have been made in addressing the issues of accessibility, affordability and adaptability of the material.

For the above reasons I have decided to release my text under an open license, even though it was published for many years through a traditional publisher.

Supporting students and instructors in a typical classroom requires much more than a textbook. Thus, while anyone is welcome to use and adapt my text at no cost, I also decided to work closely with Lyryx Learning. With colleagues at the University of Calgary, I helped create Lyryx almost 20 years ago. The original idea was to develop quality online assessment (with feedback) well beyond the multiple-choice style then available. Now Lyryx also works to provide and sustain open textbooks; working with authors, contributors, and reviewers to ensure instructors need not sacrifice quality and rigour when switching to an open text.

I believe this is the right direction for mathematical publishing going forward, and look forward to being a part of how this new approach develops.

W. Keith Nicholson, Author
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Preface

This textbook is an introduction to the ideas and techniques of linear algebra for first- or second-year students with a working knowledge of high school algebra. The contents have enough flexibility to present a traditional introduction to the subject, or to allow for a more applied course. Chapters 1–4 contain a one-semester course for beginners whereas Chapters 5–9 contain a second semester course (see the Suggested Course Outlines below). The text is primarily about real linear algebra with complex numbers being mentioned when appropriate (reviewed in Appendix A). Overall, the aim of the text is to achieve a balance among computational skills, theory, and applications of linear algebra. Calculus is not a prerequisite; places where it is mentioned may be omitted.

As a rule, students of linear algebra learn by studying examples and solving problems. Accordingly, the book contains a variety of exercises (over 1200, many with multiple parts), ordered as to their difficulty. In addition, more than 375 solved examples are included in the text, many of which are computational in nature. The examples are also used to motivate (and illustrate) concepts and theorems, carrying the student from concrete to abstract. While the treatment is rigorous, proofs are presented at a level appropriate to the student and may be omitted with no loss of continuity. As a result, the book can be used to give a course that emphasizes computation and examples, or to give a more theoretical treatment (some longer proofs are deferred to the end of the Section).

Linear Algebra has application to the natural sciences, engineering, management, and the social sciences as well as mathematics. Consequently, 18 optional “applications” sections are included in the text introducing topics as diverse as electrical networks, economic models, Markov chains, linear recurrences, systems of differential equations, and linear codes over finite fields. Additionally some applications (for example linear dynamical systems, and directed graphs) are introduced in context. The applications sections appear at the end of the relevant chapters to encourage students to browse.

SUGGESTED COURSE OUTLINES

This text includes the basis for a two-semester course in linear algebra.

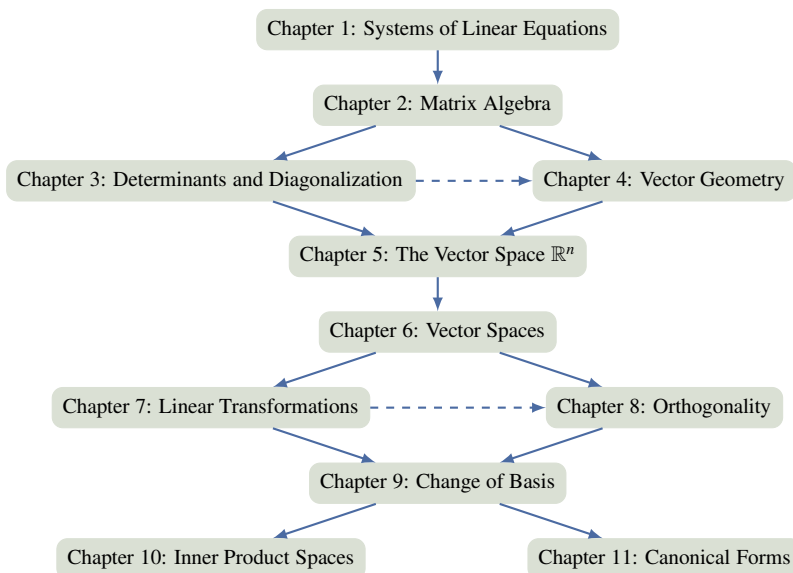
- Chapters 1–4 provide a standard one-semester course of 35 lectures, including linear equations, matrix algebra, determinants, diagonalization, and geometric vectors, with applications as time permits. At Calgary, we cover Sections 1.1–1.3, 2.1–2.6, 3.1–3.3, and 4.1–4.4 and the course is taken by all science and engineering students in their first semester. Prerequisites include a working knowledge of high school algebra (algebraic manipulations and some familiarity with polynomials); calculus is not required.
- Chapters 5–9 contain a second semester course including \mathbb{R}^n , abstract vector spaces, linear transformations (and their matrices), orthogonality, complex matrices (up to the spectral theorem) and applications. There is more material here than can be covered in one semester, and at Calgary we

cover Sections 5.1–5.5, 6.1–6.4, 7.1–7.3, 8.1–8.7, and 9.1–9.3 with a couple of applications as time permits.

- Chapter 5 is a “bridging” chapter that introduces concepts like spanning, independence, and basis in the concrete setting of \mathbb{R}^n , before venturing into the abstract in Chapter 6. The duplication is balanced by the value of reviewing these notions, and it enables the student to focus in Chapter 6 on the new idea of an abstract system. Moreover, Chapter 5 completes the discussion of rank and diagonalization from earlier chapters, and includes a brief introduction to orthogonality in \mathbb{R}^n , which creates the possibility of a one-semester, matrix-oriented course covering Chapter 1–5 for students not wanting to study the abstract theory.

CHAPTER DEPENDENCIES

The following chart suggests how the material introduced in each chapter draws on concepts covered in certain earlier chapters. A solid arrow means that ready assimilation of ideas and techniques presented in the later chapter depends on familiarity with the earlier chapter. A broken arrow indicates that some reference to the earlier chapter is made but the chapter need not be covered.



HIGHLIGHTS OF THE TEXT

- **Two-stage definition of matrix multiplication.** First, in Section 2.2 matrix-vector products are introduced naturally by viewing the left side of a system of linear equations as a product. Second, matrix-matrix products are defined in Section 2.3 by taking the columns of a product AB to be A times the corresponding columns of B . This is motivated by viewing the matrix product as composition of maps (see next item). This works well pedagogically and the usual dot-product definition follows easily. As a bonus, the proof of associativity of matrix multiplication now takes four lines.

- **Matrices as transformations.** Matrix-column multiplications are viewed (in Section 2.2) as transformations $\mathbb{R}^n \rightarrow \mathbb{R}^m$. These maps are then used to describe simple geometric reflections and rotations in \mathbb{R}^2 as well as systems of linear equations.
- **Early linear transformations.** It has been said that vector spaces exist so that linear transformations can act on them—consequently these maps are a recurring theme in the text. Motivated by the matrix transformations introduced earlier, linear transformations $\mathbb{R}^n \rightarrow \mathbb{R}^m$ are defined in Section 2.6, their standard matrices are derived, and they are then used to describe rotations, reflections, projections, and other operators on \mathbb{R}^2 .
- **Early diagonalization.** As requested by engineers and scientists, this important technique is presented in the first term using only determinants and matrix inverses (before defining independence and dimension). Applications to population growth and linear recurrences are given.
- **Early dynamical systems.** These are introduced in Chapter 3, and lead (via diagonalization) to applications like the possible extinction of species. Beginning students in science and engineering can relate to this because they can see (often for the first time) the relevance of the subject to the real world.
- **Bridging chapter.** Chapter 5 lets students deal with tough concepts (like independence, spanning, and basis) in the concrete setting of \mathbb{R}^n before having to cope with abstract vector spaces in Chapter 6.
- **Examples.** The text contains over 375 worked examples, which present the main techniques of the subject, illustrate the central ideas, and are keyed to the exercises in each section.
- **Exercises.** The text contains a variety of exercises (nearly 1175, many with multiple parts), starting with computational problems and gradually progressing to more theoretical exercises. Select solutions are available at the end of the book or in the Student Solution Manual. There is a complete Solution Manual available for instructors.
- **Applications.** There are optional applications at the end of most chapters (see the list below). While some are presented in the course of the text, most appear at the end of the relevant chapter to encourage students to browse.
- **Appendices.** Because complex numbers are needed in the text, they are described in Appendix A, which includes the polar form and roots of unity. Methods of proofs are discussed in Appendix B, followed by mathematical induction in Appendix C. A brief discussion of polynomials is included in Appendix D. All these topics are presented at the high-school level.
- **Self-Study.** This text is self-contained and therefore is suitable for self-study.
- **Rigour.** Proofs are presented as clearly as possible (some at the end of the section), but they are optional and the instructor can choose how much he or she wants to prove. However the proofs are there, so this text is more rigorous than most. Linear algebra provides one of the better venues where students begin to think logically and argue concisely. To this end, there are exercises that ask the student to “show” some simple implication, and others that ask her or him to either prove a given statement or give a counterexample. I personally present a few proofs in the first semester course and more in the second (see the Suggested Course Outlines).

- **Major Theorems.** Several major results are presented in the book. Examples: Uniqueness of the reduced row-echelon form; the cofactor expansion for determinants; the Cayley-Hamilton theorem; the Jordan canonical form; Schur's theorem on block triangular form; the principal axes and spectral theorems; and others. Proofs are included because the stronger students should at least be aware of what is involved.

CHAPTER SUMMARIES

Chapter 1: Systems of Linear Equations.

A standard treatment of gaussian elimination is given. The rank of a matrix is introduced via the row-echelon form, and solutions to a homogeneous system are presented as linear combinations of basic solutions. Applications to network flows, electrical networks, and chemical reactions are provided.

Chapter 2: Matrix Algebra.

After a traditional look at matrix addition, scalar multiplication, and transposition in Section 2.1, matrix-vector multiplication is introduced in Section 2.2 by viewing the left side of a system of linear equations as the product $A\mathbf{x}$ of the coefficient matrix A with the column \mathbf{x} of variables. The usual dot-product definition of a matrix-vector multiplication follows. Section 2.2 ends by viewing an $m \times n$ matrix A as a transformation $\mathbb{R}^n \rightarrow \mathbb{R}^m$. This is illustrated for $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ by describing reflection in the x axis, rotation of \mathbb{R}^2 through $\frac{\pi}{2}$, shears, and so on.

In Section 2.3, the product of matrices A and B is defined by $AB = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_n]$, where the \mathbf{b}_i are the columns of B . A routine computation shows that this is the matrix of the transformation B followed by A . This observation is used frequently throughout the book, and leads to simple, conceptual proofs of the basic axioms of matrix algebra. Note that linearity is not required—all that is needed is some basic properties of matrix-vector multiplication developed in Section 2.2. Thus the usual arcane definition of matrix multiplication is split into two well motivated parts, each an important aspect of matrix algebra. Of course, this has the pedagogical advantage that the conceptual power of geometry can be invoked to illuminate and clarify algebraic techniques and definitions.

In Section 2.4 and 2.5 matrix inverses are characterized, their geometrical meaning is explored, and block multiplication is introduced, emphasizing those cases needed later in the book. Elementary matrices are discussed, and the Smith normal form is derived. Then in Section 2.6, linear transformations $\mathbb{R}^n \rightarrow \mathbb{R}^m$ are defined and shown to be matrix transformations. The matrices of reflections, rotations, and projections in the plane are determined. Finally, matrix multiplication is related to directed graphs, matrix LU-factorization is introduced, and applications to economic models and Markov chains are presented.

Chapter 3: Determinants and Diagonalization.

The cofactor expansion is stated (proved by induction later) and used to define determinants inductively and to deduce the basic rules. The product and adjugate theorems are proved. Then the diagonalization algorithm is presented (motivated by an example about the possible extinction of a species of birds). As requested by our Engineering Faculty, this is done earlier than in most texts because it requires only determinants and matrix inverses, avoiding any need for subspaces, independence and dimension. Eigenvectors of a 2×2 matrix A are described geometrically (using the A -invariance of lines through the origin). Diagonalization is then used to study discrete linear dynamical systems and to discuss applications to linear recurrences and systems of differential equations. A brief discussion of Google PageRank is included.

Chapter 4: Vector Geometry.

Vectors are presented intrinsically in terms of length and direction, and are related to matrices via coordinates. Then vector operations are defined using matrices and shown to be the same as the corresponding intrinsic definitions. Next, dot products and projections are introduced to solve problems about lines and planes. This leads to the cross product. Then matrix transformations are introduced in \mathbb{R}^3 , matrices of projections and reflections are derived, and areas and volumes are computed using determinants. The chapter closes with an application to computer graphics.

Chapter 5: The Vector Space \mathbb{R}^n .

Subspaces, spanning, independence, and dimensions are introduced in the context of \mathbb{R}^n in the first two sections. Orthogonal bases are introduced and used to derive the expansion theorem. The basic properties of rank are presented and used to justify the definition given in Section 1.2. Then, after a rigorous study of diagonalization, best approximation and least squares are discussed. The chapter closes with an application to correlation and variance.

This is a “bridging” chapter, easing the transition to abstract spaces. Concern about duplication with Chapter 6 is mitigated by the fact that this is the most difficult part of the course and many students welcome a repeat discussion of concepts like independence and spanning, albeit in the abstract setting. In a different direction, Chapter 1–5 could serve as a solid introduction to linear algebra for students not requiring abstract theory.

Chapter 6: Vector Spaces.

Building on the work on \mathbb{R}^n in Chapter 5, the basic theory of abstract finite dimensional vector spaces is developed emphasizing new examples like matrices, polynomials and functions. This is the first acquaintance most students have had with an abstract system, so not having to deal with spanning, independence and dimension in the general context eases the transition to abstract thinking. Applications to polynomials and to differential equations are included.

Chapter 7: Linear Transformations.

General linear transformations are introduced, motivated by many examples from geometry, matrix theory, and calculus. Then kernels and images are defined, the dimension theorem is proved, and isomorphisms are discussed. The chapter ends with an application to linear recurrences. A proof is included that the order of a differential equation (with constant coefficients) equals the dimension of the space of solutions.

Chapter 8: Orthogonality.

The study of orthogonality in \mathbb{R}^n , begun in Chapter 5, is continued. Orthogonal complements and projections are defined and used to study orthogonal diagonalization. This leads to the principal axes theorem, the Cholesky factorization of a positive definite matrix, QR-factorization, and to a discussion of the singular value decomposition, the polar form, and the pseudoinverse. The theory is extended to \mathbb{C}^n in Section 8.7 where hermitian and unitary matrices are discussed, culminating in Schur's theorem and the spectral theorem. A short proof of the Cayley-Hamilton theorem is also presented. In Section 8.8 the field \mathbb{Z}_p of integers modulo p is constructed informally for any prime p , and codes are discussed over any finite field. The chapter concludes with applications to quadratic forms, constrained optimization, and statistical principal component analysis.

Chapter 9: Change of Basis.

The matrix of general linear transformation is defined and studied. In the case of an operator, the relationship between basis changes and similarity is revealed. This is illustrated by computing the matrix of a rotation about a line through the origin in \mathbb{R}^3 . Finally, invariant subspaces and direct sums are introduced, related to similarity, and (as an example) used to show that every involution is similar to a diagonal matrix with diagonal entries ± 1 .

Chapter 10: Inner Product Spaces.

General inner products are introduced and distance, norms, and the Cauchy-Schwarz inequality are discussed. The Gram-Schmidt algorithm is presented, projections are defined and the approximation theorem is proved (with an application to Fourier approximation). Finally, isometries are characterized, and distance preserving operators are shown to be composites of a translations and isometries.

Chapter 11: Canonical Forms.

The work in Chapter 9 is continued. Invariant subspaces and direct sums are used to derive the block triangular form. That, in turn, is used to give a compact proof of the Jordan canonical form. Of course the level is higher.

Appendices

In Appendix A, complex arithmetic is developed far enough to find n th roots. In Appendix B, methods of proof are discussed, while Appendix C presents mathematical induction. Finally, Appendix D describes the properties of polynomials in elementary terms.

LIST OF APPLICATIONS

- Network Flow (Section 1.4)
- Electrical Networks (Section 1.5)
- Chemical Reactions (Section 1.6)
- Directed Graphs (in Section 2.3)
- Input-Output Economic Models (Section 2.8)
- Markov Chains (Section 2.9)
- Polynomial Interpolation (in Section 3.2)
- Population Growth (Examples 3.3.1 and 3.3.12, Section 3.3)
- Google PageRank (in Section 3.3)
- Linear Recurrences (Section 3.4; see also Section 7.5)
- Systems of Differential Equations (Section 3.5)
- Computer Graphics (Section 4.5)
- Least Squares Approximation (in Section 5.6)
- Correlation and Variance (Section 5.7)
- Polynomials (Section 6.5)
- Differential Equations (Section 6.6)
- Linear Recurrences (Section 7.5)
- Error Correcting Codes (Section 8.8)
- Quadratic Forms (Section 8.9)
- Constrained Optimization (Section 8.10)
- Statistical Principal Component Analysis (Section 8.11)
- Fourier Approximation (Section 10.5)

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As we undertake this new publishing model with the text as an open educational resource, I would also like to thank my previous publisher. The team who supported my text greatly contributed to its success.

Now that the text has an open license, we have a much more fluid and powerful mechanism to incorporate comments and suggestions. The editorial group at Lyryx invites instructors and students to contribute to the text, and also offers to provide adaptations of the material for specific courses. Moreover the LaTeX source files are available to anyone wishing to do the adaptation and editorial work themselves!

W. Keith Nicholson

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1. Systems of Linear Equations

1.1 Solutions and Elementary Operations

Practical problems in many fields of study—such as biology, business, chemistry, computer science, economics, electronics, engineering, physics and the social sciences—can often be reduced to solving a system of linear equations. Linear algebra arose from attempts to find systematic methods for solving these systems, so it is natural to begin this book by studying linear equations.

If a , b , and c are real numbers, the graph of an equation of the form

$$ax + by = c$$

is a straight line (if a and b are not both zero), so such an equation is called a *linear* equation in the variables x and y . However, it is often convenient to write the variables as x_1, x_2, \dots, x_n , particularly when more than two variables are involved. An equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

is called a **linear equation** in the n variables x_1, x_2, \dots, x_n . Here a_1, a_2, \dots, a_n denote real numbers (called the **coefficients** of x_1, x_2, \dots, x_n , respectively) and b is also a number (called the **constant term** of the equation). A finite collection of linear equations in the variables x_1, x_2, \dots, x_n is called a **system of linear equations** in these variables. Hence,

$$2x_1 - 3x_2 + 5x_3 = 7$$

is a linear equation; the coefficients of x_1, x_2 , and x_3 are 2, -3 , and 5, and the constant term is 7. Note that each variable in a linear equation occurs to the first power only.

Given a linear equation $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$, a sequence s_1, s_2, \dots, s_n of n numbers is called a **solution** to the equation if

$$a_1s_1 + a_2s_2 + \dots + a_ns_n = b$$

that is, if the equation is satisfied when the substitutions $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ are made. A sequence of numbers is called a **solution to a system** of equations if it is a solution to every equation in the system.

For example, $x = -2, y = 5, z = 0$ and $x = 0, y = 4, z = -1$ are both solutions to the system

$$\begin{aligned}x + y + z &= 3 \\2x + y + 3z &= 1\end{aligned}$$

A system may have no solution at all, or it may have a unique solution, or it may have an infinite family of solutions. For instance, the system $x + y = 2, x + y = 3$ has no solution because the sum of two numbers cannot be 2 and 3 simultaneously. A system that has no solution is called **inconsistent**; a system with at least one solution is called **consistent**. The system in the following example has infinitely many solutions.

Example 1.1.1

Show that, for arbitrary values of s and t ,

$$\begin{aligned}x_1 &= t - s + 1 \\x_2 &= t + s + 2 \\x_3 &= s \\x_4 &= t\end{aligned}$$

is a solution to the system

$$\begin{aligned}x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\2x_1 - x_2 + 3x_3 - x_4 &= 0\end{aligned}$$

Solution. Simply substitute these values of x_1 , x_2 , x_3 , and x_4 in each equation.

$$\begin{aligned}x_1 - 2x_2 + 3x_3 + x_4 &= (t - s + 1) - 2(t + s + 2) + 3s + t = -3 \\2x_1 - x_2 + 3x_3 - x_4 &= 2(t - s + 1) - (t + s + 2) + 3s - t = 0\end{aligned}$$

Because both equations are satisfied, it is a solution for all choices of s and t .

The quantities s and t in Example 1.1.1 are called **parameters**, and the set of solutions, described in this way, is said to be given in **parametric form** and is called the **general solution** to the system. It turns out that the solutions to *every* system of equations (if there *are* solutions) can be given in parametric form (that is, the variables x_1, x_2, \dots are given in terms of new independent variables s, t , etc.). The following example shows how this happens in the simplest systems where only one equation is present.

Example 1.1.2

Describe all solutions to $3x - y + 2z = 6$ in parametric form.

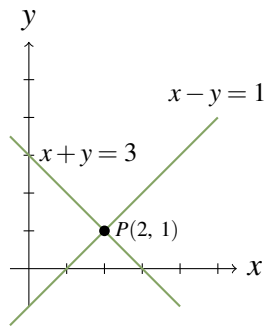
Solution. Solving the equation for y in terms of x and z , we get $y = 3x + 2z - 6$. If s and t are arbitrary then, setting $x = s$, $z = t$, we get solutions

$$\begin{aligned}x &= s \\y &= 3s + 2t - 6 \quad s \text{ and } t \text{ arbitrary} \\z &= t\end{aligned}$$

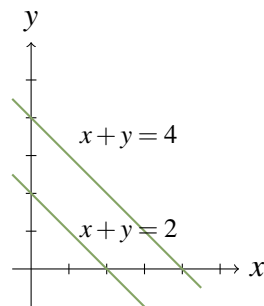
Of course we could have solved for x : $x = \frac{1}{3}(y - 2z + 6)$. Then, if we take $y = p$, $z = q$, the solutions are represented as follows:

$$\begin{aligned}x &= \frac{1}{3}(p - 2q + 6) \\y &= p \\z &= q\end{aligned} \quad p \text{ and } q \text{ arbitrary}$$

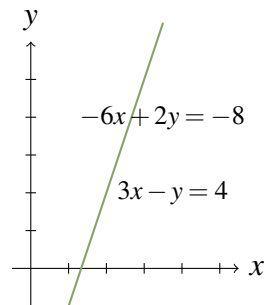
The same family of solutions can “look” quite different!



(a) Unique Solution
($x = 2, y = 1$)



(b) No Solution



(c) Infinitely many solutions
($x = t, y = 3t - 4$)

Figure 1.1.1

When only two variables are involved, the solutions to systems of linear equations can be described geometrically because the graph of a linear equation $ax + by = c$ is a straight line if a and b are not both zero. Moreover, a point $P(s, t)$ with coordinates s and t lies on the line if and only if $as + bt = c$ —that is when $x = s, y = t$ is a solution to the equation. Hence the solutions to a *system* of linear equations correspond to the points $P(s, t)$ that lie on *all* the lines in question.

In particular, if the system consists of just one equation, there must be infinitely many solutions because there are infinitely many points on a line. If the system has two equations, there are three possibilities for the corresponding straight lines:

1. *The lines intersect at a single point. Then the system has a unique solution corresponding to that point.*
2. *The lines are parallel (and distinct) and so do not intersect. Then the system has no solution.*
3. *The lines are identical. Then the system has infinitely many solutions—one for each point on the (common) line.*

These three situations are illustrated in Figure 1.1.1. In each case the graphs of two specific lines are plotted and the corresponding equations are indicated. In the last case, the equations are $3x - y = 4$ and $-6x + 2y = -8$, which have identical graphs.

With three variables, the graph of an equation $ax + by + cz = d$ can be shown to be a plane (see Section 4.2) and so again provides a “picture” of the set of solutions. However, this graphical method has its limitations: When more than three variables are involved, no physical image of the graphs (called hyperplanes) is possible. It is necessary to turn to a more “algebraic” method of solution.

Before describing the method, we introduce a concept that simplifies the computations involved. Consider the following system

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_4 &= -1 \\ 2x_1 - x_3 + 2x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 + 5x_4 &= 2 \end{aligned}$$

of three equations in four variables. The array of numbers¹

$$\left[\begin{array}{cccc|c} 3 & 2 & -1 & 1 & -1 \\ 2 & 0 & -1 & 2 & 0 \\ 3 & 1 & 2 & 5 & 2 \end{array} \right]$$

occurring in the system is called the **augmented matrix** of the system. Each row of the matrix consists of the coefficients of the variables (in order) from the corresponding equation, together with the constant

¹A rectangular array of numbers is called a **matrix**. Matrices will be discussed in more detail in Chapter 2.

term. For clarity, the constants are separated by a vertical line. The augmented matrix is just a different way of describing the system of equations. The array of coefficients of the variables

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}$$

is called the **coefficient matrix** of the system and $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ is called the **constant matrix** of the system.

Elementary Operations

The algebraic method for solving systems of linear equations is described as follows. Two such systems are said to be **equivalent** if they have the same set of solutions. A system is solved by writing a series of systems, one after the other, each equivalent to the previous system. Each of these systems has the same set of solutions as the original one; the aim is to end up with a system that is easy to solve. Each system in the series is obtained from the preceding system by a simple manipulation chosen so that it does not change the set of solutions.

As an illustration, we solve the system $x + 2y = -2$, $2x + y = 7$ in this manner. At each stage, the corresponding augmented matrix is displayed. The original system is

$$\begin{array}{l} x + 2y = -2 \\ 2x + y = 7 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 2 & 1 & 7 \end{array} \right]$$

First, subtract twice the first equation from the second. The resulting system is

$$\begin{array}{l} x + 2y = -2 \\ -3y = 11 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -3 & 11 \end{array} \right]$$

which is equivalent to the original (see Theorem 1.1.1). At this stage we obtain $y = -\frac{11}{3}$ by multiplying the second equation by $-\frac{1}{3}$. The result is the equivalent system

$$\begin{array}{l} x + 2y = -2 \\ y = -\frac{11}{3} \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -\frac{11}{3} \end{array} \right]$$

Finally, we subtract twice the second equation from the first to get another equivalent system.

$$\begin{array}{l} x = \frac{16}{3} \\ y = -\frac{11}{3} \end{array} \quad \left[\begin{array}{cc|c} 1 & 0 & \frac{16}{3} \\ 0 & 1 & -\frac{11}{3} \end{array} \right]$$

Now *this* system is easy to solve! And because it is equivalent to the original system, it provides the solution to that system.

Observe that, at each stage, a certain operation is performed on the system (and thus on the augmented matrix) to produce an equivalent system.

Definition 1.1 Elementary Operations

The following operations, called **elementary operations**, can routinely be performed on systems of linear equations to produce equivalent systems.

- I. Interchange two equations.
- II. Multiply one equation by a nonzero number.
- III. Add a multiple of one equation to a different equation.

Theorem 1.1.1

Suppose that a sequence of elementary operations is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

The proof is given at the end of this section.

Elementary operations performed on a system of equations produce corresponding manipulations of the *rows* of the augmented matrix. Thus, multiplying a row of a matrix by a number k means multiplying *every entry* of the row by k . Adding one row to another row means adding *each entry* of that row to the corresponding entry of the other row. Subtracting two rows is done similarly. Note that we regard two rows as equal when corresponding entries are the same.

In hand calculations (and in computer programs) we manipulate the rows of the augmented matrix rather than the equations. For this reason we restate these elementary operations for matrices.

Definition 1.2 Elementary Row Operations

The following are called **elementary row operations** on a matrix.

- I. Interchange two rows.
- II. Multiply one row by a nonzero number.
- III. Add a multiple of one row to a different row.

In the illustration above, a series of such operations led to a matrix of the form

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array} \right]$$

where the asterisks represent arbitrary numbers. In the case of three equations in three variables, the goal is to produce a matrix of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right]$$

This does not always happen, as we will see in the next section. Here is an example in which it does happen.

Example 1.1.3

Find all solutions to the following system of equations.

$$\begin{aligned} 3x + 4y + z &= 1 \\ 2x + 3y &= 0 \\ 4x + 3y - z &= -2 \end{aligned}$$

Solution. The augmented matrix of the original system is

$$\left[\begin{array}{ccc|c} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array} \right]$$

To create a 1 in the upper left corner we could multiply row 1 through by $\frac{1}{3}$. However, the 1 can be obtained without introducing fractions by subtracting row 2 from row 1. The result is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array} \right]$$

The upper left 1 is now used to “clean up” the first column, that is create zeros in the other positions in that column. First subtract 2 times row 1 from row 2 to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 4 & 3 & -1 & -2 \end{array} \right]$$

Next subtract 4 times row 1 from row 3. The result is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -5 & -6 \end{array} \right]$$

This completes the work on column 1. We now use the 1 in the second position of the second row to clean up the second column by subtracting row 2 from row 1 and then adding row 2 to row 3. For convenience, both row operations are done in one step. The result is

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \end{array} \right]$$

Note that the last two manipulations *did not affect* the first column (the second row has a zero there), so our previous effort there has not been undermined. Finally we clean up the third column. Begin by multiplying row 3 by $-\frac{1}{7}$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & \frac{8}{7} \end{array} \right]$$

Now subtract 3 times row 3 from row 1, and then add 2 times row 3 to row 2 to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{8}{7} \end{array} \right]$$

The corresponding equations are $x = -\frac{3}{7}$, $y = \frac{2}{7}$, and $z = \frac{8}{7}$, which give the (unique) solution.

Every elementary row operation can be **reversed** by another elementary row operation of the same type (called its **inverse**). To see how, we look at types I, II, and III separately:

Type I Interchanging two rows is reversed by interchanging them again.

Type II Multiplying a row by a nonzero number k is reversed by multiplying by $1/k$.

Type III Adding k times row p to a different row q is reversed by adding $-k$ times row p to row q (in the new matrix). Note that $p \neq q$ is essential here.

To illustrate the Type III situation, suppose there are four rows in the original matrix, denoted R_1 , R_2 , R_3 , and R_4 , and that k times R_2 is added to R_3 . Then the reverse operation adds $-k$ times R_2 to R_3 . The following diagram illustrates the effect of doing the operation first and then the reverse:

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \rightarrow \begin{bmatrix} R_1 \\ R_2 \\ R_3 + kR_2 \\ R_4 \end{bmatrix} \rightarrow \begin{bmatrix} R_1 \\ R_2 \\ (R_3 + kR_2) - kR_2 \\ R_4 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$

The existence of inverses for elementary row operations and hence for elementary operations on a system of equations, gives:

Proof of Theorem 1.1.1. Suppose that a system of linear equations is transformed into a new system by a sequence of elementary operations. Then every solution of the original system is automatically a solution of the new system because adding equations, or multiplying an equation by a nonzero number, always results in a valid equation. In the same way, each solution of the new system must be a solution to the original system because the original system can be obtained from the new one by another series of elementary operations (the inverses of the originals). It follows that the original and new systems have the same solutions. This proves Theorem 1.1.1. \square

Exercises for 1.1

Exercise 1.1.1 In each case verify that the following are solutions for all values of s and t .

a. $x = 19t - 35$
 $y = 25 - 13t$
 $z = t$

is a solution of

$$\begin{aligned} 2x + 3y + z &= 5 \\ 5x + 7y - 4z &= 0 \end{aligned}$$

b. $x_1 = 2s + 12t + 13$
 $x_2 = s$
 $x_3 = -s - 3t - 3$
 $x_4 = t$

is a solution of

$$\begin{aligned} 2x_1 + 5x_2 + 9x_3 + 3x_4 &= -1 \\ x_1 + 2x_2 + 4x_3 &= 1 \end{aligned}$$

Exercise 1.1.2 Find all solutions to the following in parametric form in two ways.

a. $3x + y = 2$ b. $2x + 3y = 1$
 c. $3x - y + 2z = 5$ d. $x - 2y + 5z = 1$

Exercise 1.1.3 Regarding $2x = 5$ as the equation $2x + 0y = 5$ in two variables, find all solutions in parametric form.

Exercise 1.1.4 Regarding $4x - 2y = 3$ as the equation $4x - 2y + 0z = 3$ in three variables, find all solutions in parametric form.

Exercise 1.1.5 Find all solutions to the general system $ax = b$ of one equation in one variable (a) when $a = 0$ and (b) when $a \neq 0$.

Exercise 1.1.6 Show that a system consisting of exactly one linear equation can have no solution, one solution, or infinitely many solutions. Give examples.

Exercise 1.1.7 Write the augmented matrix for each of the following systems of linear equations.

a. $x - 3y = 5$ b. $x + 2y = 0$
 $2x + y = 1$ $y = 1$
 c. $x - y + z = 2$ d. $x + y = 1$
 $x - z = 1$ $y + z = 0$
 $y + 2x = 0$ $z - x = 2$

Exercise 1.1.8 Write a system of linear equations that has each of the following augmented matrices.

a. $\left[\begin{array}{ccc|c} 1 & -1 & 6 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & -1 & 0 & 1 \end{array} \right]$ b. $\left[\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array} \right]$

Exercise 1.1.9 Find the solution of each of the following systems of linear equations using augmented matrices.

a. $x - 3y = 1$ b. $x + 2y = 1$
 $2x - 7y = 3$ $3x + 4y = -1$
 c. $2x + 3y = -1$ d. $3x + 4y = 1$
 $3x + 4y = 2$ $4x + 5y = -3$

Exercise 1.1.10 Find the solution of each of the following systems of linear equations using augmented matrices.

a. $x + y + 2z = -1$ b. $2x + y + z = -1$
 $2x + y + 3z = 0$ $x + 2y + z = 0$
 $-2y + z = 2$ $3x - 2z = 5$

Exercise 1.1.11 Find all solutions (if any) of the following systems of linear equations.

a. $3x - 2y = 5$ b. $3x - 2y = 5$
 $-12x + 8y = -20$ $-12x + 8y = 16$

Exercise 1.1.12 Show that the system

$$\begin{cases} x + 2y - z = a \\ 2x + y + 3z = b \\ x - 4y + 9z = c \end{cases}$$

is inconsistent unless $c = 2b - 3a$.

Exercise 1.1.13 By examining the possible positions of lines in the plane, show that two equations in two variables can have zero, one, or infinitely many solutions.

Exercise 1.1.14 In each case either show that the statement is true, or give an example² showing it is false.

- If a linear system has n variables and m equations, then the augmented matrix has n rows.
- A consistent linear system must have infinitely many solutions.
- If a row operation is done to a consistent linear system, the resulting system must be consistent.
- If a series of row operations on a linear system results in an inconsistent system, the original system is inconsistent.

Exercise 1.1.15 Find a quadratic $a + bx + cx^2$ such that the graph of $y = a + bx + cx^2$ contains each of the points $(-1, 6)$, $(2, 0)$, and $(3, 2)$.

Exercise 1.1.16 Solve the system $\begin{cases} 3x + 2y = 5 \\ 7x + 5y = 1 \end{cases}$ by changing variables $\begin{cases} x = 5x' - 2y' \\ y = -7x' + 3y' \end{cases}$ and solving the resulting equations for x' and y' .

Exercise 1.1.17 Find a , b , and c such that

$$\frac{x^2 - x + 3}{(x^2 + 2)(2x - 1)} = \frac{ax + b}{x^2 + 2} + \frac{c}{2x - 1}$$

[Hint: Multiply through by $(x^2 + 2)(2x - 1)$ and equate coefficients of powers of x .]

Exercise 1.1.18 A zookeeper wants to give an animal 42 mg of vitamin A and 65 mg of vitamin D per day. He has two supplements: the first contains 10% vitamin A and 25% vitamin D; the second contains 20% vitamin A and 25% vitamin D. How much of each supplement should he give the animal each day?

Exercise 1.1.19 Workmen John and Joe earn a total of \$24.60 when John works 2 hours and Joe works 3 hours. If John works 3 hours and Joe works 2 hours, they get \$23.90. Find their hourly rates.

Exercise 1.1.20 A biologist wants to create a diet from fish and meal containing 183 grams of protein and 93 grams of carbohydrate per day. If fish contains 70% protein and 10% carbohydrate, and meal contains 30% protein and 60% carbohydrate, how much of each food is required each day?

1.2 Gaussian Elimination

The algebraic method introduced in the preceding section can be summarized as follows: Given a system of linear equations, use a sequence of elementary row operations to carry the augmented matrix to a “nice” matrix (meaning that the corresponding equations are easy to solve). In Example 1.1.3, this nice matrix took the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right]$$

The following definitions identify the nice matrices that arise in this process.

²Such an example is called a **counterexample**. For example, if the statement is that “all philosophers have beards”, the existence of a non-bearded philosopher would be a counterexample proving that the statement is false. This is discussed again in Appendix B.

Definition 1.3 Row-Echelon Form (Reduced)

A matrix is said to be in **row-echelon form** (and will be called a **row-echelon matrix**) if it satisfies the following three conditions:

1. All **zero rows** (consisting entirely of zeros) are at the bottom.
2. The first nonzero entry from the left in each nonzero row is a 1, called the **leading 1** for that row.
3. Each leading 1 is to the right of all leading 1s in the rows above it.

A row-echelon matrix is said to be in **reduced row-echelon form** (and will be called a **reduced row-echelon matrix**) if, in addition, it satisfies the following condition:

4. Each leading 1 is the only nonzero entry in its column.

The row-echelon matrices have a “staircase” form, as indicated by the following example (the asterisks indicate arbitrary numbers).

$$\begin{bmatrix} 0 & \boxed{1} & * & * & * & * & * \\ 0 & 0 & \boxed{0} & \boxed{1} & * & * & * \\ 0 & 0 & 0 & 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1s proceed “down and to the right” through the matrix. Entries above and to the right of the leading 1s are arbitrary, but all entries below and to the left of them are zero. Hence, a matrix in row-echelon form is in reduced form if, in addition, the entries directly above each leading 1 are all zero. Note that a matrix in row-echelon form can, with a few more row operations, be carried to reduced form (use row operations to create zeros above each leading one in succession, beginning from the right).

Example 1.2.1

The following matrices are in row-echelon form (for any choice of numbers in *-positions).

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

The following, on the other hand, are in reduced row-echelon form.

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The choice of the positions for the leading 1s determines the (reduced) row-echelon form (apart from the numbers in *-positions).

The importance of row-echelon matrices comes from the following theorem.

Theorem 1.2.1

Every matrix can be brought to (reduced) row-echelon form by a sequence of elementary row operations.

In fact we can give a step-by-step procedure for actually finding a row-echelon matrix. Observe that while there are many sequences of row operations that will bring a matrix to row-echelon form, the one we use is systematic and is easy to program on a computer. Note that the algorithm deals with matrices in general, possibly with columns of zeros.

Gaussian³Algorithm⁴

Step 1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form.

Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it a), and move the row containing that entry to the top position.

Step 3. Now multiply the new top row by $1/a$ to create a leading 1.

Step 4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

This completes the first row, and all further row operations are carried out on the remaining rows.

Step 5. Repeat steps 1–4 on the matrix consisting of the remaining rows.

The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.

Observe that the gaussian algorithm is recursive: When the first leading 1 has been obtained, the procedure is repeated on the remaining rows of the matrix. This makes the algorithm easy to use on a computer. Note that the solution to Example 1.1.3 did not use the gaussian algorithm as written because the first leading 1 was not created by dividing row 1 by 3. The reason for this is that it avoids fractions. However, the general pattern is clear: Create the leading 1s from left to right, using each of them in turn to create zeros below it. Here are two more examples.

³Carl Friedrich Gauss (1777–1855) ranks with Archimedes and Newton as one of the three greatest mathematicians of all time. He was a child prodigy and, at the age of 21, he gave the first proof that every polynomial has a complex root. In 1801 he published a timeless masterpiece, *Disquisitiones Arithmeticae*, in which he founded modern number theory. He went on to make ground-breaking contributions to nearly every branch of mathematics, often well before others rediscovered and published the results.

⁴The algorithm was known to the ancient Chinese.

Example 1.2.2

Solve the following system of equations.

$$\begin{aligned} 3x + y - 4z &= -1 \\ x + 10z &= 5 \\ 4x + y + 6z &= 1 \end{aligned}$$

Solution. The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 1 & -4 & -1 \\ 1 & 0 & 10 & 5 \\ 4 & 1 & 6 & 1 \end{array} \right]$$

Create the first leading one by interchanging rows 1 and 2

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & 1 & 6 & 1 \end{array} \right]$$

Now subtract 3 times row 1 from row 2, and subtract 4 times row 1 from row 3. The result is

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 1 & -34 & -19 \end{array} \right]$$

Now subtract row 2 from row 3 to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & -3 \end{array} \right]$$

This means that the following reduced system of equations

$$\begin{aligned} x + 10z &= 5 \\ y - 34z &= -16 \\ 0 &= -3 \end{aligned}$$

is equivalent to the original system. In other words, the two have the same solutions. But this last system clearly has no solution (the last equation requires that x , y and z satisfy $0x + 0y + 0z = -3$, and no such numbers exist). Hence the original system has no solution.

Example 1.2.3

Solve the following system of equations.

$$\begin{aligned}x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\2x_1 - 4x_2 + x_3 &= 5 \\x_1 - 2x_2 + 2x_3 - 3x_4 &= 4\end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array} \right]$$

Subtracting twice row 1 from row 2 and subtracting row 1 from row 3 gives

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array} \right]$$

Now subtract row 2 from row 3 and multiply row 2 by $\frac{1}{3}$ to get

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

This is in row-echelon form, and we take it to reduced form by adding row 2 to row 1:

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The corresponding reduced system of equations is

$$\begin{aligned}x_1 - 2x_2 + x_4 &= 2 \\x_3 - 2x_4 &= 1 \\0 &= 0\end{aligned}$$

The leading ones are in columns 1 and 3 here, so the corresponding variables x_1 and x_3 are called leading variables. Because the matrix is in reduced row-echelon form, these equations can be used to solve for the leading variables in terms of the nonleading variables x_2 and x_4 . More precisely, in the present example we set $x_2 = s$ and $x_4 = t$ where s and t are arbitrary, so these equations become

$$x_1 - 2s + t = 2 \quad \text{and} \quad x_3 - 2t = 1$$

Finally the solutions are given by

$$\begin{aligned}x_1 &= 2 + 2s - t \\x_2 &= s \\x_3 &= 1 + 2t \\x_4 &= t\end{aligned}$$

where s and t are arbitrary.

The solution of Example 1.2.3 is typical of the general case. To solve a linear system, the augmented matrix is carried to reduced row-echelon form, and the variables corresponding to the leading ones are called **leading variables**. Because the matrix is in reduced form, each leading variable occurs in exactly one equation, so that equation can be solved to give a formula for the leading variable in terms of the nonleading variables. It is customary to call the nonleading variables “free” variables, and to label them by new variables s, t, \dots , called **parameters**. Hence, as in Example 1.2.3, every variable x_i is given by a formula in terms of the parameters s and t . Moreover, every choice of these parameters leads to a solution to the system, and every solution arises in this way. This procedure works in general, and has come to be called

Gaussian Elimination

To solve a system of linear equations proceed as follows:

1. Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations.
2. If a row $[0 \ 0 \ 0 \ \cdots \ 0 \ 1]$ occurs, the system is inconsistent.
3. Otherwise, assign the nonleading variables (if any) as parameters, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters.

There is a variant of this procedure, wherein the augmented matrix is carried only to row-echelon form. The nonleading variables are assigned as parameters as before. Then the last equation (corresponding to the row-echelon form) is used to solve for the last leading variable in terms of the parameters. This last leading variable is then substituted into all the preceding equations. Then, the second last equation yields the second last leading variable, which is also substituted back. The process continues to give the general solution. This procedure is called **back-substitution**. This procedure can be shown to be numerically more efficient and so is important when solving very large systems.⁵

Example 1.2.4

Find a condition on the numbers a, b , and c such that the following system of equations is consistent. When that condition is satisfied, find all solutions (in terms of a, b , and c).

$$\begin{aligned}x_1 + 3x_2 + x_3 &= a \\ -x_1 - 2x_2 + x_3 &= b \\ 3x_1 + 7x_2 - x_3 &= c\end{aligned}$$

Solution. We use gaussian elimination except that now the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array} \right]$$

⁵With n equations where n is large, gaussian elimination requires roughly $n^3/2$ multiplications and divisions, whereas this number is roughly $n^3/3$ if back substitution is used.

has entries a , b , and c as well as known numbers. The first leading one is in place, so we create zeros below it in column 1:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & c-3a \end{array} \right]$$

The second leading 1 has appeared, so use it to create zeros in the rest of column 2:

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -2a-3b \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & c-a+2b \end{array} \right]$$

Now the whole solution depends on the number $c - a + 2b = c - (a - 2b)$. The last row corresponds to an equation $0 = c - (a - 2b)$. If $c \neq a - 2b$, there is *no* solution (just as in Example 1.2.2). Hence:

The system is consistent if and only if $c = a - 2b$.

In this case the last matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -2a-3b \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Thus, if $c = a - 2b$, taking $x_3 = t$ where t is a parameter gives the solutions

$$x_1 = 5t - (2a + 3b) \quad x_2 = (a + b) - 2t \quad x_3 = t.$$

Rank

It can be proven that the *reduced* row-echelon form of a matrix A is uniquely determined by A . That is, no matter which series of row operations is used to carry A to a reduced row-echelon matrix, the result will always be the same matrix. (A proof is given at the end of Section 2.5.) By contrast, this is not true for row-echelon matrices: Different series of row operations can carry the same matrix A to *different* row-echelon matrices. Indeed, the matrix $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -1 & 2 \end{bmatrix}$ can be carried (by one row operation) to the row-echelon matrix $\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -6 \end{bmatrix}$, and then by another row operation to the (reduced) row-echelon matrix $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -6 \end{bmatrix}$. However, it *is* true that the number r of leading 1s must be the same in each of these row-echelon matrices (this will be proved in Chapter 5). Hence, the number r depends only on A and not on the way in which A is carried to row-echelon form.

Definition 1.4 Rank of a Matrix

The **rank** of matrix A is the number of leading 1s in any row-echelon matrix to which A can be carried by row operations.

Example 1.2.5

Compute the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$.

Solution. The reduction of A to row-echelon form is

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & -8 \\ 0 & 1 & -5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because this row-echelon matrix has two leading 1s, $\text{rank } A = 2$.

Suppose that $\text{rank } A = r$, where A is a matrix with m rows and n columns. Then $r \leq m$ because the leading 1s lie in different rows, and $r \leq n$ because the leading 1s lie in different columns. Moreover, the rank has a useful application to equations. Recall that a system of linear equations is called consistent if it has at least one solution.

Theorem 1.2.2

Suppose a system of m equations in n variables is **consistent**, and that the rank of the augmented matrix is r .

1. The set of solutions involves exactly $n - r$ parameters.
2. If $r < n$, the system has infinitely many solutions.
3. If $r = n$, the system has a unique solution.

Proof. The fact that the rank of the augmented matrix is r means there are exactly r leading variables, and hence exactly $n - r$ nonleading variables. These nonleading variables are all assigned as parameters in the gaussian algorithm, so the set of solutions involves exactly $n - r$ parameters. Hence if $r < n$, there is at least one parameter, and so infinitely many solutions. If $r = n$, there are no parameters and so a unique solution. \square

Theorem 1.2.2 shows that, for any system of linear equations, exactly three possibilities exist:

1. *No solution.* This occurs when a row $[0 \ 0 \ \cdots \ 0 \ 1]$ occurs in the row-echelon form. This is the case where the system is inconsistent.
2. *Unique solution.* This occurs when every variable is a leading variable.

3. *Infinitely many solutions.* This occurs when the system is consistent and there is at least one nonleading variable, so at least one parameter is involved.

Example 1.2.6

Suppose the matrix A in Example 1.2.5 is the augmented matrix of a system of $m = 3$ linear equations in $n = 3$ variables. As $\text{rank } A = r = 2$, the set of solutions will have $n - r = 1$ parameter. The reader can verify this fact directly.

Many important problems involve **linear inequalities** rather than **linear equations**. For example, a condition on the variables x and y might take the form of an inequality $2x - 5y \leq 4$ rather than an equality $2x - 5y = 4$. There is a technique (called the **simplex algorithm**) for finding solutions to a system of such inequalities that maximizes a function of the form $p = ax + by$ where a and b are fixed constants.

Exercises for 1.2

Exercise 1.2.1 Which of the following matrices are in reduced row-echelon form? Which are in row-echelon form?

a.
$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b.
$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c.
$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d.
$$\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

e.
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

f.
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 1.2.2 Carry each of the following matrices to reduced row-echelon form.

a.
$$\begin{bmatrix} 0 & -1 & 2 & 1 & 2 & 1 & -1 \\ 0 & 1 & -2 & 2 & 7 & 2 & 4 \\ 0 & -2 & 4 & 3 & 7 & 1 & 0 \\ 0 & 3 & -6 & 1 & 6 & 4 & 1 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & -1 & 3 & 1 & 3 & 2 & 1 \\ 0 & -2 & 6 & 1 & -5 & 0 & -1 \\ 0 & 3 & -9 & 2 & 4 & 1 & -1 \\ 0 & 1 & -3 & -1 & 3 & 0 & 1 \end{bmatrix}$$

Exercise 1.2.3 The augmented matrix of a system of linear equations has been carried to the following by row operations. In each case solve the system.

a.
$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

b.
$$\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

c.
$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & 3 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

d.
$$\left[\begin{array}{cccccc|c} 1 & -1 & 2 & 4 & 6 & 2 & 2 \\ 0 & 1 & 2 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Exercise 1.2.4 Find all solutions (if any) to each of the following systems of linear equations.

a.
$$\begin{aligned} x - 2y &= 1 \\ 4y - x &= -2 \end{aligned}$$

b.
$$\begin{aligned} 3x - y &= 0 \\ 2x - 3y &= 1 \end{aligned}$$

$$\begin{aligned} \text{c. } 2x + y &= 5 \\ 3x + 2y &= 6 \end{aligned}$$

$$\begin{aligned} \text{e. } 3x - y &= 4 \\ 2y - 6x &= 1 \end{aligned}$$

$$\begin{aligned} \text{d. } 3x - y &= 2 \\ 2y - 6x &= -4 \end{aligned}$$

$$\begin{aligned} \text{f. } 2x - 3y &= 5 \\ 3y - 2x &= 2 \end{aligned}$$

$$\begin{aligned} \text{d. } x_1 + x_2 + 2x_3 - x_4 &= 4 \\ 3x_2 - x_3 + 4x_4 &= 2 \\ x_1 + 2x_2 - 3x_3 + 5x_4 &= 0 \\ x_1 + x_2 - 5x_3 + 6x_4 &= -3 \end{aligned}$$

Exercise 1.2.5 Find all solutions (if any) to each of the following systems of linear equations.

$$\begin{aligned} \text{a. } x + y + 2z &= 8 \\ 3x - y + z &= 0 \\ -x + 3y + 4z &= -4 \end{aligned}$$

$$\begin{aligned} \text{c. } x + y - z &= 10 \\ -x + 4y + 5z &= -5 \\ x + 6y + 3z &= 15 \end{aligned}$$

$$\begin{aligned} \text{e. } 5x + y &= 2 \\ 3x - y + 2z &= 1 \\ x + y - z &= 5 \end{aligned}$$

$$\begin{aligned} \text{g. } x + y + z &= 2 \\ x + z &= 1 \\ 2x + 5y + 2z &= 7 \end{aligned}$$

$$\begin{aligned} \text{b. } -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$$

$$\begin{aligned} \text{d. } x + 2y - z &= 2 \\ 2x + 5y - 3z &= 1 \\ x + 4y - 3z &= 3 \end{aligned}$$

$$\begin{aligned} \text{f. } 3x - 2y + z &= -2 \\ x - y + 3z &= 5 \\ -x + y + z &= -1 \end{aligned}$$

$$\begin{aligned} \text{h. } x + 2y - 4z &= 10 \\ 2x - y + 2z &= 5 \\ x + y - 2z &= 7 \end{aligned}$$

Exercise 1.2.6 Express the last equation of each system as a sum of multiples of the first two equations. [Hint: Label the equations, use the gaussian algorithm.]

$$\begin{aligned} \text{a. } x_1 + x_2 + x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 3 \\ x_1 - 2x_2 + 2x_3 &= 2 \end{aligned}$$

$$\begin{aligned} \text{b. } x_1 + 2x_2 - 3x_3 &= -3 \\ x_1 + 3x_2 - 5x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -35 \end{aligned}$$

Exercise 1.2.7 Find all solutions to the following systems.

$$\begin{aligned} \text{a. } 3x_1 + 8x_2 - 3x_3 - 14x_4 &= 2 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 1 \\ x_1 - 2x_2 + x_3 + 10x_4 &= 0 \\ x_1 + 5x_2 - 2x_3 - 12x_4 &= 1 \end{aligned}$$

$$\begin{aligned} \text{b. } x_1 - x_2 + x_3 - x_4 &= 0 \\ -x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{aligned} \text{c. } x_1 - x_2 + x_3 - 2x_4 &= 1 \\ -x_1 + x_2 + x_3 + x_4 &= -1 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 2 \\ x_1 - x_2 + 2x_3 + x_4 &= 1 \end{aligned}$$

Exercise 1.2.8 In each of the following, find (if possible) conditions on a and b such that the system has no solution, one solution, and infinitely many solutions.

$$\begin{aligned} \text{a. } x - 2y &= 1 \\ ax + by &= 5 \end{aligned}$$

$$\begin{aligned} \text{c. } x - by &= -1 \\ x + ay &= 3 \end{aligned}$$

$$\begin{aligned} \text{b. } x + by &= -1 \\ ax + 2y &= 5 \end{aligned}$$

$$\begin{aligned} \text{d. } ax + y &= 1 \\ 2x + y &= b \end{aligned}$$

Exercise 1.2.9 In each of the following, find (if possible) conditions on a , b , and c such that the system has no solution, one solution, or infinitely many solutions.

$$\begin{aligned} \text{a. } 3x + y - z &= a \\ x - y + 2z &= b \\ 5x + 3y - 4z &= c \end{aligned}$$

$$\begin{aligned} \text{c. } -x + 3y + 2z &= -8 \\ x + z &= 2 \\ 3x + 3y + az &= b \end{aligned}$$

$$\begin{aligned} \text{e. } 3x - y + 2z &= 3 \\ x + y - z &= 2 \\ 2x - 2y + 3z &= b \end{aligned}$$

$$\begin{aligned} \text{f. } x + ay - z &= 1 \\ -x + (a-2)y + z &= -1 \\ 2x + 2y + (a-2)z &= 1 \end{aligned}$$

$$\begin{aligned} \text{b. } 2x + y - z &= a \\ 2y + 3z &= b \\ x - z &= c \end{aligned}$$

$$\begin{aligned} \text{d. } x + ay &= 0 \\ y + bz &= 0 \\ z + cx &= 0 \end{aligned}$$

Exercise 1.2.10 Find the rank of each of the matrices in Exercise 1.2.1.

Exercise 1.2.11 Find the rank of each of the following matrices.

$$\text{a. } \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$\text{c. } \begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix}$$

$$\text{e. } \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 1 & 2-a & -1 & -2a^2 \end{bmatrix}$$

$$\text{f. } \begin{bmatrix} 1 & 1 & 2 & a^2 \\ 1 & 1-a & 2 & 0 \\ 2 & 2-a & 6-a & 4 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} -2 & 3 & 3 \\ 3 & -4 & 1 \\ -5 & 7 & 2 \end{bmatrix}$$

$$\text{d. } \begin{bmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

Exercise 1.2.12 Consider a system of linear equations with augmented matrix A and coefficient matrix C . In each case either prove the statement or give an example showing that it is false.

- If there is more than one solution, A has a row of zeros.
- If A has a row of zeros, there is more than one solution.
- If there is no solution, the reduced row-echelon form of C has a row of zeros.
- If the row-echelon form of C has a row of zeros, there is no solution.
- There is no system that is inconsistent for every choice of constants.
- If the system is consistent for some choice of constants, it is consistent for every choice of constants.

Now assume that the augmented matrix A has 3 rows and 5 columns.

- If the system is consistent, there is more than one solution.
- The rank of A is at most 3.
- If $\text{rank } A = 3$, the system is consistent.
- If $\text{rank } C = 3$, the system is consistent.

Exercise 1.2.13 Find a sequence of row operations carrying

$$\begin{bmatrix} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ c_1 + a_1 & c_2 + a_2 & c_3 + a_3 \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \end{bmatrix} \text{ to } \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Exercise 1.2.14 In each case, show that the reduced row-echelon form is as given.

- $\begin{bmatrix} p & 0 & a \\ b & 0 & 0 \\ q & c & r \end{bmatrix}$ with $abc \neq 0$; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}$ where $c \neq a$ or $b \neq a$;
 $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$

Exercise 1.2.15 Show that $\begin{cases} az + by + cz = 0 \\ a_1x + b_1y + c_1z = 0 \end{cases}$ always has a solution other than $x = 0, y = 0, z = 0$.

Exercise 1.2.16 Find the circle $x^2 + y^2 + ax + by + c = 0$ passing through the following points.

- $(-2, 1), (5, 0),$ and $(4, 1)$
- $(1, 1), (5, -3),$ and $(-3, -3)$

Exercise 1.2.17 Three Nissans, two Fords, and four Chevrolets can be rented for \$106 per day. At the same rates two Nissans, four Fords, and three Chevrolets cost \$107 per day, whereas four Nissans, three Fords, and two Chevrolets cost \$102 per day. Find the rental rates for all three kinds of cars.

Exercise 1.2.18 A school has three clubs and each student is required to belong to exactly one club. One year the students switched club membership as follows:

Club A. $\frac{4}{10}$ remain in A, $\frac{1}{10}$ switch to B, $\frac{5}{10}$ switch to C.

Club B. $\frac{7}{10}$ remain in B, $\frac{2}{10}$ switch to A, $\frac{1}{10}$ switch to C.

Club C. $\frac{6}{10}$ remain in C, $\frac{2}{10}$ switch to A, $\frac{2}{10}$ switch to B.

If the fraction of the student population in each club is unchanged, find each of these fractions.

Exercise 1.2.19 Given points $(p_1, q_1), (p_2, q_2),$ and (p_3, q_3) in the plane with $p_1, p_2,$ and p_3 distinct, show that they lie on some curve with equation $y = a + bx + cx^2$. [Hint: Solve for $a, b,$ and c .]

Exercise 1.2.20 The scores of three players in a tournament have been lost. The only information available is the total of the scores for players 1 and 2, the total for players 2 and 3, and the total for players 3 and 1.

- Show that the individual scores can be rediscovered.
- Is this possible with four players (knowing the totals for players 1 and 2, 2 and 3, 3 and 4, and 4 and 1)?

Exercise 1.2.21 A boy finds \$1.05 in dimes, nickels, and pennies. If there are 17 coins in all, how many coins of each type can he have?

Exercise 1.2.22 If a consistent system has more variables than equations, show that it has infinitely many solutions. [Hint: Use Theorem 1.2.2.]

1.3 Homogeneous Equations

A system of equations in the variables x_1, x_2, \dots, x_n is called **homogeneous** if all the constant terms are zero—that is, if each equation of the system has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Clearly $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is a solution to such a system; it is called the **trivial solution**. Any solution in which at least one variable has a nonzero value is called a **nontrivial solution**. Our chief goal in this section is to give a useful condition for a homogeneous system to have nontrivial solutions. The following example is instructive.

Example 1.3.1

Show that the following homogeneous system has nontrivial solutions.

$$\begin{aligned}x_1 - x_2 + 2x_3 - x_4 &= 0 \\2x_1 + 2x_2 + x_4 &= 0 \\3x_1 + x_2 + 2x_3 - x_4 &= 0\end{aligned}$$

Solution. The reduction of the augmented matrix to reduced row-echelon form is outlined below.

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 0 & 4 & -4 & 3 & 0 \\ 0 & 4 & -4 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

The leading variables are $x_1, x_2,$ and x_4 , so x_3 is assigned as a parameter—say $x_3 = t$. Then the general solution is $x_1 = -t, x_2 = t, x_3 = t, x_4 = 0$. Hence, taking $t = 1$ (say), we get a nontrivial solution: $x_1 = -1, x_2 = 1, x_3 = 1, x_4 = 0$.

The existence of a nontrivial solution in Example 1.3.1 is ensured by the presence of a parameter in the solution. This is due to the fact that there is a *nonleading* variable (x_3 in this case). But there *must* be a nonleading variable here because there are four variables and only three equations (and hence at *most* three leading variables). This discussion generalizes to a proof of the following fundamental theorem.

Theorem 1.3.1

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Proof. Suppose there are m equations in n variables where $n > m$, and let R denote the reduced row-echelon form of the augmented matrix. If there are r leading variables, there are $n - r$ nonleading variables, and so $n - r$ parameters. Hence, it suffices to show that $r < n$. But $r \leq m$ because R has r leading 1s and m rows, and $m < n$ by hypothesis. So $r \leq m < n$, which gives $r < n$. \square

Note that the converse of Theorem 1.3.1 is not true: if a homogeneous system has nontrivial solutions, it need not have more variables than equations (the system $x_1 + x_2 = 0$, $2x_1 + 2x_2 = 0$ has nontrivial solutions but $m = 2 = n$.)

Theorem 1.3.1 is very useful in applications. The next example provides an illustration from geometry.

Example 1.3.2

We call the graph of an equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ a **conic** if the numbers a , b , and c are not all zero. Show that there is at least one conic through any five points in the plane that are not all on a line.

Solution. Let the coordinates of the five points be (p_1, q_1) , (p_2, q_2) , (p_3, q_3) , (p_4, q_4) , and (p_5, q_5) . The graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ passes through (p_i, q_i) if

$$ap_i^2 + bp_iq_i + cq_i^2 + dp_i + eq_i + f = 0$$

This gives five equations, one for each i , linear in the six variables a , b , c , d , e , and f . Hence, there is a nontrivial solution by Theorem 1.3.1. If $a = b = c = 0$, the five points all lie on the line with equation $dx + ey + f = 0$, contrary to assumption. Hence, one of a , b , c is nonzero.

Linear Combinations and Basic Solutions

As for rows, two columns are regarded as **equal** if they have the same number of entries and corresponding entries are the same. Let \mathbf{x} and \mathbf{y} be columns with the same number of entries. As for elementary row operations, their **sum** $\mathbf{x} + \mathbf{y}$ is obtained by adding corresponding entries and, if k is a number, the **scalar product** $k\mathbf{x}$ is defined by multiplying each entry of \mathbf{x} by k . More precisely:

$$\text{If } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ then } \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \text{ and } k\mathbf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix}.$$

A sum of scalar multiples of several columns is called a **linear combination** of these columns. For example, $s\mathbf{x} + t\mathbf{y}$ is a linear combination of \mathbf{x} and \mathbf{y} for any choice of numbers s and t .

Example 1.3.3

$$\text{If } \mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ then } 2\mathbf{x} + 5\mathbf{y} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Example 1.3.4

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. If $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, determine whether \mathbf{v} and \mathbf{w} are linear combinations of \mathbf{x} , \mathbf{y} and \mathbf{z} .

Solution. For \mathbf{v} , we must determine whether numbers r , s , and t exist such that $\mathbf{v} = r\mathbf{x} + s\mathbf{y} + t\mathbf{z}$, that is, whether

$$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r+2s+3t \\ s+t \\ r+t \end{bmatrix}$$

Equating corresponding entries gives a system of linear equations $r+2s+3t=0$, $s+t=-1$, and $r+t=2$ for r , s , and t . By gaussian elimination, the solution is $r=2-k$, $s=-1-k$, and $t=k$ where k is a parameter. Taking $k=0$, we see that $\mathbf{v} = 2\mathbf{x} - \mathbf{y}$ is a linear combination of \mathbf{x} , \mathbf{y} , and \mathbf{z} . Turning to \mathbf{w} , we again look for r , s , and t such that $\mathbf{w} = r\mathbf{x} + s\mathbf{y} + t\mathbf{z}$; that is,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r+2s+3t \\ s+t \\ r+t \end{bmatrix}$$

leading to equations $r+2s+3t=1$, $s+t=1$, and $r+t=1$ for real numbers r , s , and t . But this time there is *no* solution as the reader can verify, so \mathbf{w} is *not* a linear combination of \mathbf{x} , \mathbf{y} , and \mathbf{z} .

Our interest in linear combinations comes from the fact that they provide one of the best ways to describe the general solution of a homogeneous system of linear equations. When solving such a system

with n variables x_1, x_2, \dots, x_n , write the variables as a column⁶ matrix: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. The trivial solution

is denoted $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. As an illustration, the general solution in Example 1.3.1 is $x_1 = -t$, $x_2 = t$, $x_3 = t$,

and $x_4 = 0$, where t is a parameter, and we would now express this by saying that the general solution is $\mathbf{x} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix}$, where t is arbitrary.

Now let \mathbf{x} and \mathbf{y} be two solutions to a homogeneous system with n variables. Then any linear combination $s\mathbf{x} + t\mathbf{y}$ of these solutions turns out to be again a solution to the system. More generally:

$$\text{Any linear combination of solutions to a homogeneous system is again a solution.} \quad (1.1)$$

⁶The reason for using columns will be apparent later.

In fact, suppose that a typical equation in the system is $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, and suppose that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ are solutions. Then } a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \text{ and } a_1y_1 + a_2y_2 + \cdots + a_ny_n = 0.$$

$$\text{Hence } s\mathbf{x} + t\mathbf{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \\ \vdots \\ sx_n + ty_n \end{bmatrix} \text{ is also a solution because}$$

$$\begin{aligned} & a_1(sx_1 + ty_1) + a_2(sx_2 + ty_2) + \cdots + a_n(sx_n + ty_n) \\ &= [a_1(sx_1) + a_2(sx_2) + \cdots + a_n(sx_n)] + [a_1(ty_1) + a_2(ty_2) + \cdots + a_n(ty_n)] \\ &= s(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + t(a_1y_1 + a_2y_2 + \cdots + a_ny_n) \\ &= s(0) + t(0) \\ &= 0 \end{aligned}$$

A similar argument shows that Statement 1.1 is true for linear combinations of more than two solutions.

The remarkable thing is that *every* solution to a homogeneous system is a linear combination of certain particular solutions and, in fact, these solutions are easily computed using the gaussian algorithm. Here is an example.

Example 1.3.5

Solve the homogeneous system with coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

Solution. The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 0 \\ -2 & 4 & 4 & -2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

so the solutions are $x_1 = 2s + \frac{1}{5}t$, $x_2 = s$, $x_3 = \frac{3}{5}t$, and $x_4 = t$ by gaussian elimination. Hence we can write the general solution \mathbf{x} in the matrix form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s + \frac{1}{5}t \\ s \\ \frac{3}{5}t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = s\mathbf{x}_1 + t\mathbf{x}_2.$$

Here $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix}$ are particular solutions determined by the gaussian algorithm.

The solutions \mathbf{x}_1 and \mathbf{x}_2 in Example 1.3.5 are denoted as follows:

Definition 1.5 Basic Solutions

*The gaussian algorithm systematically produces solutions to any homogeneous linear system, called **basic solutions**, one for every parameter.*

Moreover, the algorithm gives a routine way to express *every* solution as a linear combination of basic solutions as in Example 1.3.5, where the general solution \mathbf{x} becomes

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{5}t \begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \end{bmatrix}$$

Hence by introducing a new parameter $r = t/5$ we can multiply the original basic solution \mathbf{x}_2 by 5 and so eliminate fractions. For this reason:

Convention:

Any nonzero scalar multiple of a basic solution will still be called a basic solution.

In the same way, the gaussian algorithm produces basic solutions to *every* homogeneous system, one for each parameter (there are *no* basic solutions if the system has only the trivial solution). Moreover every solution is given by the algorithm as a linear combination of these basic solutions (as in Example 1.3.5). If A has rank r , Theorem 1.2.2 shows that there are exactly $n - r$ parameters, and so $n - r$ basic solutions. This proves:

Theorem 1.3.2

Let A be an $m \times n$ matrix of rank r , and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. *The system has exactly $n - r$ basic solutions, one for each parameter.*
2. *Every solution is a linear combination of these basic solutions.*

Example 1.3.6

Find basic solutions of the homogeneous system with coefficient matrix A , and express every solution as a linear combination of the basic solutions, where

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

Solution. The reduction of the augmented matrix to reduced row-echelon form is

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & 2 & 2 & 0 \\ -2 & 6 & 1 & 2 & -5 & 0 \\ 3 & -9 & -1 & 0 & 7 & 0 \\ -3 & 9 & 2 & 6 & -8 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

so the general solution is $x_1 = 3r - 2s - 2t$, $x_2 = r$, $x_3 = -6s + t$, $x_4 = s$, and $x_5 = t$ where r , s , and t are parameters. In matrix form this is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 2s - 2t \\ r \\ -6s + t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence basic solutions are

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Exercises for 1.3

Exercise 1.3.1 Consider the following statements about a system of linear equations with augmented matrix A . In each case either prove the statement or give an example for which it is false.

- If the system is homogeneous, every solution is trivial.
- If the system has a nontrivial solution, it cannot be homogeneous.
- If there exists a trivial solution, the system is homogeneous.
- If the system is consistent, it must be homogeneous.

Now assume that the system is homogeneous.

- If there exists a nontrivial solution, there is no trivial solution.
- If there exists a solution, there are infinitely many solutions.
- If there exist nontrivial solutions, the row-echelon form of A has a row of zeros.
- If the row-echelon form of A has a row of zeros, there exist nontrivial solutions.
- If a row operation is applied to the system, the new system is also homogeneous.

Exercise 1.3.2 In each of the following, find all values of a for which the system has nontrivial solutions, and determine all solutions in each case.

- | | |
|---------------------|---------------------|
| a. $x - 2y + z = 0$ | b. $x + 2y + z = 0$ |
| $x + ay - 3z = 0$ | $x + 3y + 6z = 0$ |
| $-x + 6y - 5z = 0$ | $2x + 3y + az = 0$ |
| c. $x + y - z = 0$ | d. $ax + y + z = 0$ |
| $ay - z = 0$ | $x + y - z = 0$ |
| $x + y + az = 0$ | $x + y + az = 0$ |

Exercise 1.3.3 Let $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and

$\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. In each case, either write \mathbf{v} as a linear combination of \mathbf{x} , \mathbf{y} , and \mathbf{z} , or show that it is not such a linear combination.

- | | |
|--|--|
| a. $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ | b. $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix}$ |
| c. $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ | d. $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$ |

Exercise 1.3.4 In each case, either express \mathbf{y} as a linear combination of \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 , or show that it is not such a linear combination. Here:

$$\mathbf{a}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- | | |
|--|---|
| a. $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}$ | b. $\mathbf{y} = \begin{bmatrix} -1 \\ 9 \\ 2 \\ 6 \end{bmatrix}$ |
|--|---|

Exercise 1.3.5 For each of the following homogeneous systems, find a set of basic solutions and express the general solution as a linear combination of these basic solutions.

- $$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 + 2x_3 + x_5 &= 0 \\ 2x_1 + 4x_2 - 2x_3 + 3x_4 + x_5 &= 0 \end{aligned}$$
- $$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\ -x_1 - 2x_2 + 2x_3 + x_5 &= 0 \\ -x_1 - 2x_2 + 3x_3 + x_4 + 3x_5 &= 0 \end{aligned}$$
- $$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\ 2x_1 + 3x_2 - x_3 + 2x_4 + x_5 &= 0 \\ 4x_1 + 5x_2 - 2x_3 + 5x_4 + 2x_5 &= 0 \end{aligned}$$

$$\begin{aligned} \text{d. } & x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 = 0 \\ & 2x_1 + 2x_2 - 4x_3 - 4x_4 + x_5 = 0 \\ & x_1 - x_2 + 2x_3 + 4x_4 + x_5 = 0 \\ & -2x_1 - 4x_2 + 8x_3 + 10x_4 + x_5 = 0 \end{aligned}$$

Exercise 1.3.6

- a. Does Theorem 1.3.1 imply that the system $\begin{cases} -z + 3y = 0 \\ 2x - 6y = 0 \end{cases}$ has nontrivial solutions? Explain.
- b. Show that the converse to Theorem 1.3.1 is not true. That is, show that the existence of nontrivial solutions does *not* imply that there are more variables than equations.

Exercise 1.3.7 In each case determine how many solutions (and how many parameters) are possible for a homogeneous system of four linear equations in six variables with augmented matrix A . Assume that A has nonzero entries. Give all possibilities.

- a. Rank $A = 2$. b. Rank $A = 1$.
- c. A has a row of zeros.
- d. The row-echelon form of A has a row of zeros.

Exercise 1.3.8 The graph of an equation $ax + by + cz = 0$ is a plane through the origin (provided that not all of a , b , and c are zero). Use Theorem 1.3.1 to show that two planes through the origin have a point in common other than the origin $(0, 0, 0)$.

Exercise 1.3.9

- a. Show that there is a line through any pair of points in the plane. [*Hint*: Every line has equation $ax + by + c = 0$, where a , b , and c are not all zero.]
- b. Generalize and show that there is a plane $ax + by + cz + d = 0$ through any three points in space.

Exercise 1.3.10 The graph of

$$a(x^2 + y^2) + bx + cy + d = 0$$

is a circle if $a \neq 0$. Show that there is a circle through any three points in the plane that are not all on a line.

Exercise 1.3.11 Consider a homogeneous system of linear equations in n variables, and suppose that the augmented matrix has rank r . Show that the system has nontrivial solutions if and only if $n > r$.

Exercise 1.3.12 If a consistent (possibly nonhomogeneous) system of linear equations has more variables than equations, prove that it has more than one solution.

1.4 An Application to Network Flow

There are many types of problems that concern a network of conductors along which some sort of flow is observed. Examples of these include an irrigation network and a network of streets or freeways. There are often points in the system at which a net flow either enters or leaves the system. The basic principle behind the analysis of such systems is that the total flow into the system must equal the total flow out. In fact, we apply this principle at every junction in the system.

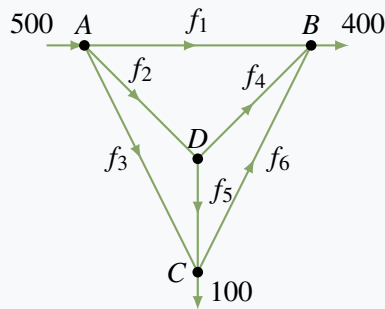
Junction Rule

At each of the junctions in the network, the total flow into that junction must equal the total flow out.

This requirement gives a linear equation relating the flows in conductors emanating from the junction.

Example 1.4.1

A network of one-way streets is shown in the accompanying diagram. The rate of flow of cars into intersection A is 500 cars per hour, and 400 and 100 cars per hour emerge from B and C , respectively. Find the possible flows along each street.



Solution. Suppose the flows along the streets are f_1, f_2, f_3, f_4, f_5 , and f_6 cars per hour in the directions shown. Then, equating the flow in with the flow out at each intersection, we get

$$\begin{array}{ll} \text{Intersection A} & 500 = f_1 + f_2 + f_3 \\ \text{Intersection B} & f_1 + f_4 + f_6 = 400 \\ \text{Intersection C} & f_3 + f_5 = f_6 + 100 \\ \text{Intersection D} & f_2 = f_4 + f_5 \end{array}$$

These give four equations in the six variables f_1, f_2, \dots, f_6 .

$$\begin{array}{rcl} f_1 + f_2 + f_3 & & = 500 \\ f_1 & + f_4 & + f_6 = 400 \\ & f_3 & + f_5 - f_6 = 100 \\ f_2 & - f_4 - f_5 & = 0 \end{array}$$

The reduction of the augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 500 \\ 1 & 0 & 0 & 1 & 0 & 1 & 400 \\ 0 & 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 400 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Hence, when we use f_4, f_5 , and f_6 as parameters, the general solution is

$$f_1 = 400 - f_4 - f_6 \quad f_2 = f_4 + f_5 \quad f_3 = 100 - f_5 + f_6$$

This gives all solutions to the system of equations and hence all the possible flows.

Of course, not all these solutions may be acceptable in the real situation. For example, the flows f_1, f_2, \dots, f_6 are all *positive* in the present context (if one came out negative, it would mean traffic flowed in the opposite direction). This imposes constraints on the flows: $f_1 \geq 0$ and $f_3 \geq 0$ become

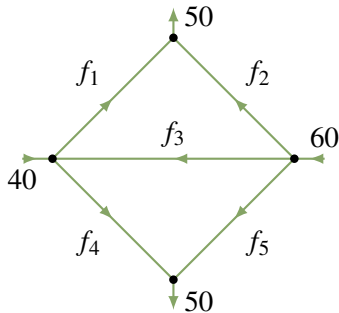
$$f_4 + f_6 \leq 400 \quad f_5 - f_6 \leq 100$$

Further constraints might be imposed by insisting on maximum values on the flow in each street.

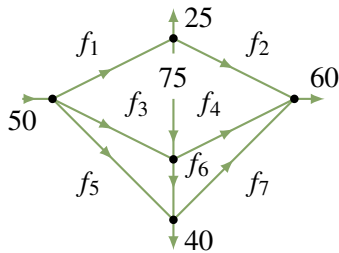
Exercises for 1.4

Exercise 1.4.1 Find the possible flows in each of the following networks of pipes.

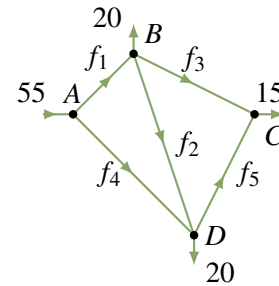
a.



b.



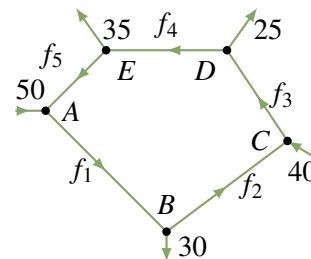
Exercise 1.4.2 A proposed network of irrigation canals is described in the accompanying diagram. At peak demand, the flows at interchanges A , B , C , and D are as shown.



a. Find the possible flows.

b. If canal BC is closed, what range of flow on AD must be maintained so that no canal carries a flow of more than 30?

Exercise 1.4.3 A traffic circle has five one-way streets, and vehicles enter and leave as shown in the accompanying diagram.



a. Compute the possible flows.

b. Which road has the heaviest flow?

1.5 An Application to Electrical Networks⁷

In an electrical network it is often necessary to find the current in amperes (A) flowing in various parts of the network. These networks usually contain resistors that retard the current. The resistors are indicated by a symbol ($\sim\sim\sim$), and the resistance is measured in ohms (Ω). Also, the current is increased at various points by voltage sources (for example, a battery). The voltage of these sources is measured in volts (V),

⁷This section is independent of Section 1.4

and they are represented by the symbol $(-\overrightarrow{|})$. We assume these voltage sources have no resistance. The flow of current is governed by the following principles.

Ohm's Law

The current I and the voltage drop V across a resistance R are related by the equation $V = RI$.

Kirchhoff's Laws

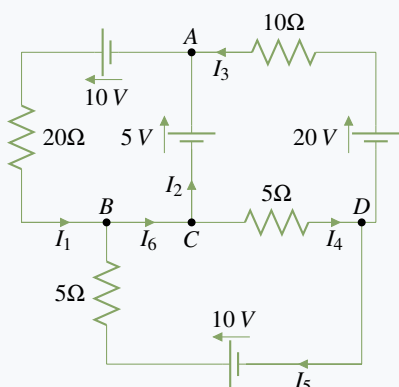
1. (Junction Rule) The current flow into a junction equals the current flow out of that junction.
2. (Circuit Rule) The algebraic sum of the voltage drops (due to resistances) around any closed circuit of the network must equal the sum of the voltage increases around the circuit.

When applying rule 2, select a direction (clockwise or counterclockwise) around the closed circuit and then consider all voltages and currents positive when in this direction and negative when in the opposite direction. This is why the term *algebraic sum* is used in rule 2. Here is an example.

Example 1.5.1

Find the various currents in the circuit shown.

Solution.



First apply the junction rule at junctions A , B , C , and D to obtain

$$\begin{array}{ll} \text{Junction } A & I_1 = I_2 + I_3 \\ \text{Junction } B & I_6 = I_1 + I_5 \\ \text{Junction } C & I_2 + I_4 = I_6 \\ \text{Junction } D & I_3 + I_5 = I_4 \end{array}$$

Note that these equations are not independent (in fact, the third is an easy consequence of the other three). Next, the circuit rule insists that the sum of the voltage increases (due to the sources) around a closed circuit must equal the sum of the voltage drops (due to resistances). By Ohm's law, the voltage

loss across a resistance R (in the direction of the current I) is RI . Going counterclockwise around three closed circuits yields

$$\begin{array}{ll} \text{Upper left} & 10 + 5 = 20I_1 \\ \text{Upper right} & -5 + 20 = 10I_3 + 5I_4 \\ \text{Lower} & -10 = -20I_5 - 5I_4 \end{array}$$

Hence, disregarding the redundant equation obtained at junction C , we have six equations in the six unknowns I_1, \dots, I_6 . The solution is

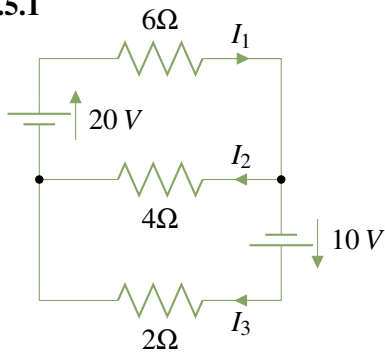
$$\begin{aligned}
 I_1 &= \frac{15}{20} & I_4 &= \frac{28}{20} \\
 I_2 &= -\frac{1}{20} & I_5 &= \frac{12}{20} \\
 I_3 &= \frac{16}{20} & I_6 &= \frac{27}{20}
 \end{aligned}$$

The fact that I_2 is negative means, of course, that this current is in the opposite direction, with a magnitude of $\frac{1}{20}$ amperes.

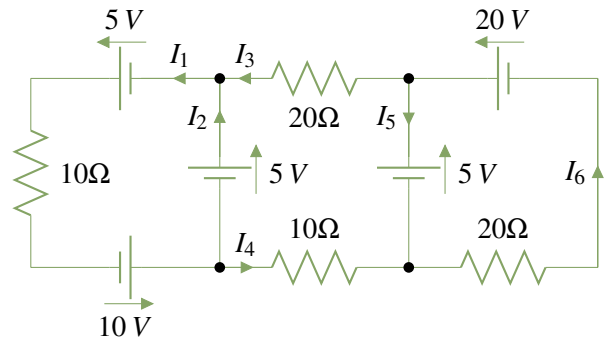
Exercises for 1.5

In Exercises 1 to 4, find the currents in the circuits.

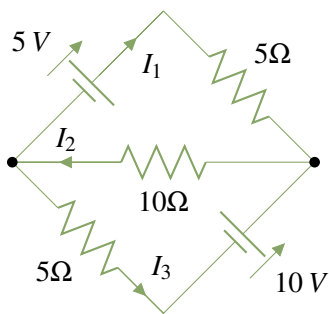
Exercise 1.5.1



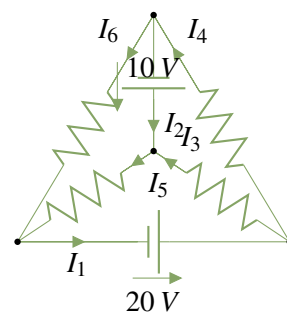
Exercise 1.5.3



Exercise 1.5.2

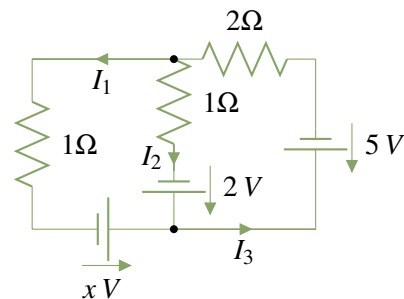


Exercise 1.5.4 All resistances are 10Ω.



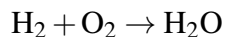
Exercise 1.5.5

Find the voltage x such that the current $I_1 = 0$.

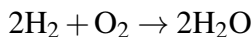


1.6 An Application to Chemical Reactions

When a chemical reaction takes place a number of molecules combine to produce new molecules. Hence, when hydrogen H_2 and oxygen O_2 molecules combine, the result is water H_2O . We express this as



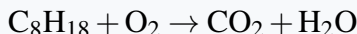
Individual atoms are neither created nor destroyed, so the number of hydrogen and oxygen atoms going into the reaction must equal the number coming out (in the form of water). In this case the reaction is said to be *balanced*. Note that each hydrogen molecule H_2 consists of two atoms as does each oxygen molecule O_2 , while a water molecule H_2O consists of two hydrogen atoms and one oxygen atom. In the above reaction, this requires that twice as many hydrogen molecules enter the reaction; we express this as follows:



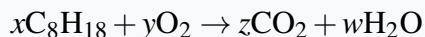
This is now balanced because there are 4 hydrogen atoms and 2 oxygen atoms on each side of the reaction.

Example 1.6.1

Balance the following reaction for burning octane C_8H_{18} in oxygen O_2 :



where CO_2 represents carbon dioxide. We must find positive integers x , y , z , and w such that



Equating the number of carbon, hydrogen, and oxygen atoms on each side gives $8x = z$, $18x = 2w$ and $2y = 2z + w$, respectively. These can be written as a homogeneous linear system

$$\begin{aligned} 8x - z &= 0 \\ 18x - 2w &= 0 \\ 2y - 2z - w &= 0 \end{aligned}$$

which can be solved by gaussian elimination. In larger systems this is necessary but, in such a simple situation, it is easier to solve directly. Set $w = t$, so that $x = \frac{1}{9}t$, $z = \frac{8}{9}t$, $2y = \frac{16}{9}t + t = \frac{25}{9}t$. But x , y , z , and w must be positive integers, so the smallest value of t that eliminates fractions is 18. Hence, $x = 2$, $y = 25$, $z = 16$, and $w = 18$, and the balanced reaction is



The reader can verify that this is indeed balanced.

It is worth noting that this problem introduces a new element into the theory of linear equations: the insistence that the solution must consist of positive integers.

Exercises for 1.6

In each case balance the chemical reaction.

Exercise 1.6.1 $\text{CH}_4 + \text{O}_2 \rightarrow \text{CO}_2 + \text{H}_2\text{O}$. This is the burning of methane CH_4 .

Exercise 1.6.2 $\text{NH}_3 + \text{CuO} \rightarrow \text{N}_2 + \text{Cu} + \text{H}_2\text{O}$. Here NH_3 is ammonia, CuO is copper oxide, Cu is copper, and N_2 is nitrogen.

Exercise 1.6.3 $\text{CO}_2 + \text{H}_2\text{O} \rightarrow \text{C}_6\text{H}_{12}\text{O}_6 + \text{O}_2$. This is called the photosynthesis reaction— $\text{C}_6\text{H}_{12}\text{O}_6$ is glucose.

Exercise 1.6.4 $\text{Pb}(\text{N}_3)_2 + \text{Cr}(\text{MnO}_4)_2 \rightarrow \text{Cr}_2\text{O}_3 + \text{MnO}_2 + \text{Pb}_3\text{O}_4 + \text{NO}$.

Supplementary Exercises for Chapter 1

Exercise 1.1 We show in Chapter 4 that the graph of an equation $ax + by + cz = d$ is a plane in space when not all of a , b , and c are zero.

- By examining the possible positions of planes in space, show that three equations in three variables can have zero, one, or infinitely many solutions.
- Can two equations in three variables have a unique solution? Give reasons for your answer.

Exercise 1.2 Find all solutions to the following systems of linear equations.

$$\begin{aligned} \text{a. } & x_1 + x_2 + x_3 - x_4 = 3 \\ & 3x_1 + 5x_2 - 2x_3 + x_4 = 1 \\ & -3x_1 - 7x_2 + 7x_3 - 5x_4 = 7 \\ & x_1 + 3x_2 - 4x_3 + 3x_4 = -5 \end{aligned}$$

$$\begin{aligned} \text{b. } & x_1 + 4x_2 - x_3 + x_4 = 2 \\ & 3x_1 + 2x_2 + x_3 + 2x_4 = 5 \\ & x_1 - 6x_2 + 3x_3 = 1 \\ & x_1 + 14x_2 - 5x_3 + 2x_4 = 3 \end{aligned}$$

Exercise 1.3 In each case find (if possible) conditions on a , b , and c such that the system has zero, one, or infinitely many solutions.

$$\begin{array}{ll} \text{a. } & x + 2y - 4z = 4 \\ & 3x - y + 13z = 2 \\ & 4x + y + a^2z = a + 3 \end{array} \quad \begin{array}{l} \text{b. } \\ \\ \end{array} \begin{array}{l} x + y + 3z = a \\ ax + y + 5z = 4 \\ x + ay + 4z = a \end{array}$$

Exercise 1.4 Show that any two rows of a matrix can be interchanged by elementary row transformations of the other two types.

Exercise 1.5 If $ad \neq bc$, show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has reduced row-echelon form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Exercise 1.6 Find a , b , and c so that the system

$$\begin{aligned} x + ay + cz &= 0 \\ bx + cy - 3z &= 1 \\ ax + 2y + bz &= 5 \end{aligned}$$

has the solution $x = 3$, $y = -1$, $z = 2$.

Exercise 1.7 Solve the system

$$\begin{aligned} x + 2y + 2z &= -3 \\ 2x + y + z &= -4 \\ x - y + iz &= i \end{aligned}$$

where $i^2 = -1$. [See Appendix A.]

Exercise 1.8 Show that the *real* system

$$\begin{cases} x + y + z = 5 \\ 2x - y - z = 1 \\ -3x + 2y + 2z = 0 \end{cases}$$

has a *complex* solution: $x = 2$, $y = i$, $z = 3 - i$ where $i^2 = -1$. Explain. What happens when such a real system has a unique solution?

Exercise 1.9 A man is ordered by his doctor to take 5 units of vitamin A, 13 units of vitamin B, and 23 units of vitamin C each day. Three brands of vitamin pills are available, and the number of units of each vitamin per pill are shown in the accompanying table.

Brand	Vitamin		
	A	B	C
1	1	2	4
2	1	1	3
3	0	1	1

- Find all combinations of pills that provide exactly the required amount of vitamins (no partial pills allowed).
- If brands 1, 2, and 3 cost 3¢, 2¢, and 5¢ per pill, respectively, find the least expensive treatment.

Exercise 1.10 A restaurant owner plans to use x tables seating 4, y tables seating 6, and z tables seating 8, for a total of 20 tables. When fully occupied, the tables seat 108 customers. If only half of the x tables, half of the y tables, and one-fourth of the z tables are used, each fully occupied, then 46 customers will be seated. Find x , y , and z .

Exercise 1.11

- Show that a matrix with two rows and two columns that is in reduced row-echelon form must have one of the following forms:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$$

[Hint: The leading 1 in the first row must be in column 1 or 2 or not exist.]

- List the seven reduced row-echelon forms for matrices with two rows and three columns.
- List the four reduced row-echelon forms for matrices with three rows and two columns.

Exercise 1.12 An amusement park charges \$7 for adults, \$2 for youths, and \$0.50 for children. If 150 people enter and pay a total of \$100, find the numbers of adults, youths, and children. [Hint: These numbers are nonnegative integers.]

Exercise 1.13 Solve the following system of equations for x and y .

$$\begin{aligned} x^2 + xy - y^2 &= 1 \\ 2x^2 - xy + 3y^2 &= 13 \\ x^2 + 3xy + 2y^2 &= 0 \end{aligned}$$

[Hint: These equations are linear in the new variables $x_1 = x^2$, $x_2 = xy$, and $x_3 = y^2$.]