Quiz 3 Review

Lec 12

Q Let W be the subspace of vectors in $\mathbb{R}^3$ whose 3rd entry is 0.

- Is $S := \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ a spanning set of W?

No, S is not a subset of W

- Is $S := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ a spanning set of W?

Yes. Computation: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 3 & b \\ 0 & 0 & 0 & 0 \end{array}\right]$ No Leading 1 in right col means the system is Consistent

$\underbrace{}_{\text{already in REF}}$

So there is a solution for $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ for all $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ in W.

- Is S above linearly independent?

No, $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ has infinitely many solutions

- Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ a basis for W?

Yes. Computation: $\left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 3 & b \\ 0 & 0 & 0 \end{array}\right] \rightarrow \underbrace{\left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b/3 \\ 0 & 0 & 0 \end{array}\right]}_{\text{REF}}$

There is one unique linear combination

$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ for all $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ in W.

Lec 13a

Def

A <u>basis</u> of a subspace W of $\mathbb{R}^n$ is a set S of vectors in W where

- S is linearly independent
- S is a spanning set of S

Q: Let W be the subspace of $\mathbb{R}^3$ consisting of vectors whose entries sum to 0.

- Is $S := \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ a basis for W? No, they are not even in W

- Is $S := \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ a basis for W? No, they are not linearly independent.

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

- Is $S := \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ a basis for W?

<u>Ans</u> $$\left[\begin{array}{cc|c} 1 & 0 & a \\ -1 & 1 & b \\ 0 & -1 & -a-b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & a+b \\ 0 & -1 & -a-b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & a+b \\ 0 & 0 & 0 \end{array}\right]$$

yes, every elt in W is a unique lin. combi of S.

<u>Q</u>.

- The <u>standard basis for $\mathbb{R}^3$</u> is... $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

- Is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^2$?

Yes, $\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right) \neq 0$.

- Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$? No, too few (must have 3 vectors to be a basis)

Lec 13b (Basis Algo for im(A))

Q: Let $A := \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Find a basis for im(A)

<u>Ans</u>

Put A in REF

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Take the original 1st, 3rd, 4th cols of A:

A basis for im(A) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Q: rank(A)?
dim(im(A))?

Ans is 3

Lec 14 a (Basis Algo for the kernel of a matrix)

Q : Let $A := \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Find a basis for $\ker(A)$.

Ans

Put A in REF

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Write a general solution to $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Since 2nd col has no leading 1, set $\boxed{x_2 = t}$

$$\left.\begin{aligned} x_1 - 2x_2 &= 0 \\ x_3 - 6x_4 &= 0 \\ \boxed{x_4 = 0} \end{aligned}\right\} \Rightarrow \quad x_1 - 2t = 0 \Rightarrow \boxed{x_1 = 2t}, \quad \boxed{x_3 = 0}$$

Gen. sol : $\begin{bmatrix} 2t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

A basis for $\ker(A)$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Lec 14 b

Subspaces of $\mathbb{R}^4$ whose dimension is easy to compute

- $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ has dim 0

- If $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is not the zero vector, the subspace

$$\left\{ r \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; r \text{ in } \mathbb{R} \right\} = \operatorname{span}\left(\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\} \right) \quad \text{has dim } 1$$

- If V is a subspace of $\mathbb{R}^4$ with dim 4, then V must be the entire $\mathbb{R}^4$.

Q: Let $A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

- rank(A)? $\cancel{0}$ A is not the zero matrix $\quad \cancel{1}$ Rows of A are not multiple of each other $\quad 2$

- dim(im(A))? (same as rank(A) = 2)

- dim(ker(A))? $\underbrace{\text{width}(A)}_{2} = \underbrace{\text{rank}(A)}_{2} + \underbrace{\dim(\ker(A))}_{\text{must be } 0}$. $\quad \boxed{\text{Rank-Nullity Thm}}$

- Describe ker(A). Ans $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, the zero subspace

$\boxed{\text{Lec 15a}}$

Def If A is an $n \times n$ matrix, an eigenbasis of A is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Q: Let $A := \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$. Let $S := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Do S form an eigenbasis for A?

Ans ① Check that the vectors in S are eigenvectors

- $\underbrace{\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ✓

- $A \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ ✓

- $A \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ ✓

② Check that S is a basis for $\mathbb{R}^3$

You can check that det(concatenation) $\neq 0$

or check rank(concatenation) $= 3$

or check concatenation is invertible.

I'll check this

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{\text{Concatenation}} \xrightarrow[\text{swap } R_2, R_3]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow[R_3 \mapsto 2R_2 + R_3]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so rank(Concatenation of S) $= 3$

$\therefore$ Yes, S is an eigenbasis of A.

Lec 15 b

- Algorithm to find an eigenbasis for an $n\times n$ matrix A or show A does not have an eigenbasis.

① Find all eigenvalues λ of A

② For each λ, find a basis of the λ-eigenspace of A
$= \ker(A-\lambda Id)$

③ Put all vectors from part ② together.
- If there are n vectors, the set is an eigenbasis
- If there are fewer than n, A has no eigenbasis.

Q: Let $A := \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix}$ be a 3×3 matrix

① with eigenvalues 1 and 7.

② dimension of $\text{im}(A-7Id) = 1$

Does A have an eigenbasis?

Answer ② • Rank-nullity Thm says: $\text{width}(M) = \dim(\text{im}(M)) + \dim(\ker(M))$
$3 = \text{width}(A-7Id) = 1 + \dim(\ker(A-7Id))$

So dimension of the 7-eigenspace of A is $3-1=2$.
A basis for this 7-eigenspace of A has two vectors.

- dimension of the 1-eigenspace of A is at least 1 (since it contains a 1-eigenvector of A).

Step 3: • We have two eigenvectors from the basis of the 7-eigenspace and at least one eigenvector from the basis of the 1-eigenspace,
so A has an eigenbasis

Lec 15 c

Def / Fact A matrix M with an eigenbasis is diagonalizable, i.e. we can write $M = PDP^{-1}$ where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ is a diagonal matrix w/ $\lambda_1, \dots, \lambda_n$ eigenvalues of A and P is a concatenation of the eigenbasis (the order has to match $\lambda_1, \dots, \lambda_n$).

Q: A is a 5×5 matrix with eigenvalues $0, 1, 2, 5, 6$.

- Does A have an eigenbasis? Ans Yes, each λ-eigenspace contributes one eigenvector
- Is A diagonalizable? Ans Yes, since A has an eigenbasis.

Q: Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. I computed an eigenbasis of A:

$$\underset{\lambda_1 = 3}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \quad \underset{\lambda_2 = 2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \quad \underset{\lambda_3 = 1}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}$$

- Write a diagonalization of A
- Write A^{100} using this diagonalization.

Answer • Let $P := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. Then $A = P \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} P^{-1}$.

- $$\begin{aligned} A^{100} &= (PDP^{-1})^{100} \\ &= PDP^{-1} PDP^{-1} PDP^{-1} \dots PDP^{-1} \\ &= PD^{100}P^{-1} \\ &= P \begin{bmatrix} 3^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}. \end{aligned}$$

Q: Let $M = \begin{bmatrix} 1 & 3 & 7 & 0 \\ 3 & 0 & 6 & 0 \\ 7 & 6 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Does M have an eigenbasis? Is M diagonalizable?

Ans Yes, by spectral Thm (since M is symmetric: $M^T = M$).

Lec 16

- <u>Idea</u>

A **vector space** is a set V in which ...

- there is a rule to add any two elements in V
- there is a rule to multiply any elt in V by a number

and both operations "behave like" vector addition & vector scalar multiplication.

Note: Every vector space has a "zero" element.

- Examples of vector spaces
  - $\mathbb{R}^n$ vectors of height n
  - $\mathbb{P}$ all polynomials in x
  - $\mathbb{P}_5$ all polynomials of degree at most 5
  - C^∞ all smooth functions
  - Solutions to the differential equation $f'' = -f$
  - Solutions to the differential equation $f' = f$
  - Any subspace of a vector space is itself a vector space (with respect to the same addition and scalar multp rule)

Q: Is $x^3 + 5x$ a scalar multiple of $x^2 + 5$?

<u>Answer</u> Is there a number c where $x^3 + 5x = c(x^2+5)$?

$\boxed{\text{No!}}$

Q: Write x^2 as a linear combination of $1+2x+x^2$, $1+x$, and 1.

<u>Answer</u> Find c_1, c_2, c_3 so that $x^2 = c_1(1+2x+x^2) + c_2(1+x) + c_3(1)$.

Rewrite in standard form: $x^2 + 0x + 0.1 = (c_1)x^2 + (2c_1 + c_2)x + (c_1 + c_2 + c_3)$

$$\left.\begin{aligned} c_1 &= \boxed{1} \\ 2c_1 + c_2 &= 0 \\ c_1 + c_2 + c_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= \boxed{-2} \\ c_3 &= -1+2 = \boxed{1} \end{aligned}$$

So $\boxed{x^2 = \boxed{1}(1+2x+x^2) + \boxed{-2}(1+x) + \boxed{1}(1).}$

Con't Lec 16

Examples of subspaces of $\mathbb{P}$ (and therefore examples of vector spaces):

- $\mathbb{P}_5$
- Set of polynomials with a factor of $(x+1)$,
  $\{(x+1)f \mid f \text{ in } \mathbb{P}\}$
- Set of polynomials which has -1 as a zero,
  $\{f \text{ in } \mathbb{P} \mid f(-1)=0\}$

Note: these two describe the same set!

Examples of subsets of $\mathbb{P}$ that are not subspaces

- Set of polynomials of degree exactly 5
  Why? The zero polynomial is not in this set
- Set of polynomials $f(x)$ where $f(-1)=6$.
  Why? The zero polynomial is not in this set

- Examples of subspaces of C^∞ (hence examples of vector spaces)

- The set of solutions to $f''=-f$