9b

Let $A := \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$

Domain of T_A ? $\mathbb{R}^2$ — # of cols

Target of T_A ? $\mathbb{R}^3$ — # of rows / height of A

Find a vector v so that $T_A(v) = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

Use the fact that

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] \xrightarrow[\text{reduce}]{\text{Row}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Answer: $y = 2,\ x = -1 \Rightarrow v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Check $A \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+4 \\ -4+10 \\ -7+16 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ ✓

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Def Saying that a function $f: \underset{(\text{domain})}{\mathbb{R}^n} \to \underset{(\text{target})}{\mathbb{R}^m}$ preserves addition means ...

Answer: For all v, w in $\mathbb{R}^n$, we have

$$f(v+w) = f(v) + f(w).$$

Which transformations are linear transformations?

- ☐ $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 + y \\ 2y \end{bmatrix}$ No, not of the form $\begin{bmatrix} a_1 x + b_1 y \\ a_2 x + b_2 y \\ \vdots \end{bmatrix}$
- ☐ $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + y \\ 2y \\ 4x \end{bmatrix}$ yes, $f = T_A$ for $A := \begin{bmatrix} 5 & 1 \\ 0 & 2 \\ 4 & 0 \end{bmatrix}$ because $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$, $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
- ☐ $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2 \\ y \end{bmatrix}$ No, not of the form $\begin{bmatrix} a_1 x + b_1 y \\ a_2 x + b_2 y \\ \vdots \end{bmatrix}$ Also, we've seen that a translation is not a linear transformation

- ☐ A rotation by 50° around the origin
- ☐ A reflection with respect to the line $y = 4x$
- ☐ A projection onto the line $y = 4x$

} yes

(See Exercise 2 Lec 10a)

Suppose $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $f(v_1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $f(v_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

and $v_3 = 3v_1 - 2v_2$

Compute $f(v_3)$

Answer $f(v_3) = f(3v_1 - 2v_2)$

$= 3f(v_1) - 2f(v_2)$ because linear transformations preserve linear combinations

$= 3\begin{bmatrix} 3 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 9 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ -8 \end{bmatrix}$

$= \begin{bmatrix} 15 \\ -14 \end{bmatrix}$.

10b

What are the standard basis vectors in $\mathbb{R}^3$?

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Compute the matrix of the linear transformation

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x+y \\ 2y \\ 4x \end{bmatrix}$

$f = T_A$ for $A := \begin{bmatrix} 5 & 1 \\ 0 & 2 \\ 4 & 0 \end{bmatrix}$

because $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$, $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

11a (Write these definitions on a cheat sheet)

Def We say a subset S of $\mathbb{R}^n$ is closed under scalar multiplication if ...

If v is in S, then rv is in S for all r in $\mathbb{R}$.

Def
In this class, we defined a subspace V of $\mathbb{R}^n$ to be ...

a nonempty subset of $\mathbb{R}^n$ which is

closed under addition and

closed under scalar multiplication

If S is a subspace of $\mathbb{R}^4$, does S contain $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?

Yes (every subspace of $\mathbb{R}^n$ contains the $n \times 1$ zero vector)

Def The image of an $\overbrace{m}^{\#\text{ rows}} \times \overbrace{n}^{\#\text{cols}}$ matrix A is ...

the set $\{ v$ in $\mathbb{R}^m$ such that $v = Aw$ for some w in $\mathbb{R}^n \}$

Let $A := \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$

height of v

- Is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in $\operatorname{im}(A)$? No, $\operatorname{im}(A)$ is a subset of $\mathbb{R}^3$

- Is $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ in $\operatorname{im}(A)$? Use the fact that $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] \xrightarrow[\text{reduce}]{\text{Row}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

  Yes. There is no leading 1 in the right column of an REF augmented matrix equivalent to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$.

  This means $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is consistent.

- Is $\begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}$ in $\operatorname{im}(A)$? Use the fact that $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] \xrightarrow[\text{reduce}]{\text{Row}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

  No. There is a leading 1 in the right column of an REF augmented matrix equivalent to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$.

  This gives $0 = 1$.

  This means $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}$ is inconsistent.

- Is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in $\operatorname{im}(A)$?

  Yes. Possible reasoning:

  1. $A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
  2. $\operatorname{im}(A)$ is a subspace of $\mathbb{R}^3$.

     Every subspace contains the zero vector.
  3. $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a homogeneous linear system.

     The solution set to a homogeneous SLE contains the zero vector.