Mon + Wed Oct 5,7, 2020 Review (14 questions)

1. What are the possible number of solutions to a system of linear equations?
   - 0
   - 1
   - Infinitely many

2. If $\det(M) = 5$, how many solutions does
$$M \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
have? Why?

Exactly one. Since $\det(M) \neq 0$, M^{-1} exists.

$$M^{-1} M \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The only solution is $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

3. If B is a 6×6 matrix and $\det(B) = 5$, what is the rank of B? 6.

the max # of leading 1s of a 6×6 REF matrix is 6

Does B^{-1} exist? Yes. $\det(B) \neq 0$ if and only if B^{-1} exists.

Matrix arithmetic

4. What is the size of $\begin{bmatrix} 1 & -4 & 1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}$?

5. Explain how you compute the product $\begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 & 1 \\ -2 & 3 & 0 \end{bmatrix}$

$3 \times \cancel{2}\ \cancel{2} \times 3 \qquad 3 \times 3$

$$\begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 & 1 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -1.1+1.-2 & -1.-4+3 & -1 \\ 3.-2 & 3.3 & 0 \\ 2.1+4.-2 & 2.-4+4.3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 7 & -1 \\ -6 & 9 & 0 \\ -6 & 4 & 2 \end{bmatrix}$$

6. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

- Find A^{-1} by starting with the augmented matrix $\left[A \mid Id \right]$
- Perform a sanity check

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \mapsto 2R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Check</u> $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2-2 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

7. Find all a, b so that $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a^2 & ab + ba \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

number multp commutes

$$\left.\begin{aligned} a^2 &= 0 \\ 2ab &= 0 \\ a^2 &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} &a = 0 \\ &b \text{ is any number} \end{aligned}$$

Answer: $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for any number t

Additional Questions

8a If A is a 2x2 matrix and $A^2 - 3A + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, find a formula for A^{-1} if it exists.

Solution

$$A^2 - 3A + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= 3A - A^2 \\ &= 3A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - AA \\ &= A\left[3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - A\right] \end{aligned}$$

$$\boxed{A^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - A}$$

8b Suppose A is a 2×2 matrix and

$$A^2 - 6A + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Find A^{-1} if it exists.

<u>Solution</u>

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = -A^2 + 6A$$

$$5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -A^2 + 6A$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{5}\left[-A^2 + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A\right]$$

$$= \frac{1}{5} A \left[-A + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \quad \frac{1}{5}\left[-A + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right]$$

$$\boxed{A = \frac{1}{5}\left[-A + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right]}$$

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9. Suppose A, B, C are invertible 5×5 matrices where $A = BCBC^{-1}$

- Find a formula for the inverse of A if it exists

<u>Solution</u>

- In general, the inverse of XY is $Y^{-1}X^{-1}$
  (if X and Y are invertible)

So $A^{-1} = (C^{-1})^{-1} B^{-1} C^{-1} B^{-1}$

$= C B^{-1} C^{-1} B^{-1}$

10. Write the vector $\begin{bmatrix} a+2b \\ -a-5b+c \end{bmatrix}$ as a matrix multiplication in the form $\text{Matrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

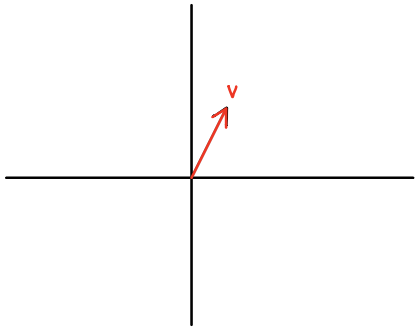
Answer:

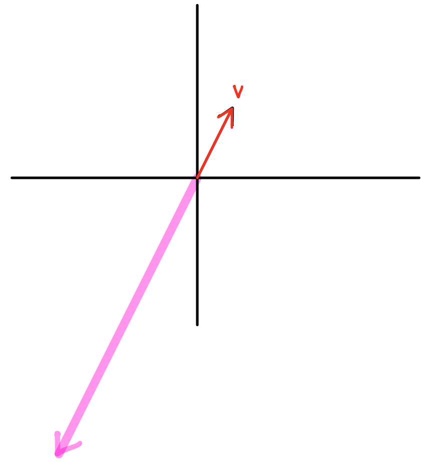
$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Check

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+2b \\ -1-5b+1 \end{bmatrix} \checkmark$$

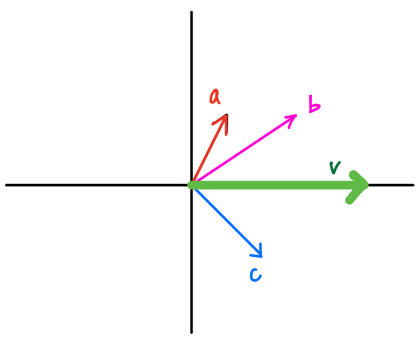
11.





This vector has length 4 times the length of v.

Write it as a scalar multiple of v.



Write the vector v as an expression using a, b, c

12) What does it mean to be an eigenvector & eigenvalue of a matrix A?

<u>Ans</u> A nonzero vector v is an eigenvector of A with eigenvalue λ (read: lambda) if

$$Av = \lambda v$$

a) Suppose $M \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

Give me an eigenvector of M and its associated eigenvalue.

a) <u>Sol</u>: $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\lambda = 0$

b) Suppose $B \begin{bmatrix}1\\0\\2\\-1\end{bmatrix} = \begin{bmatrix}5\\0\\10\\-5\end{bmatrix}$

Give me an eigenvector of B and its associated eigenvalue.

b) <u>Sol</u> $\begin{bmatrix}1\\0\\2\\-1\end{bmatrix}$, $\lambda = 5$

13) Determinants

- If $\det \begin{bmatrix}2 & 4\\3 & k\end{bmatrix} = 12$, what is k?
- Perform sanity check
- If $\det \begin{bmatrix}2 & 4\\3 & k\end{bmatrix} = 12$, is $\begin{bmatrix}2 & 4\\3 & k\end{bmatrix}$ invertible?
- If $\det \begin{bmatrix}2 & 4\\3 & k\end{bmatrix} = 12$, what is the rank of $\begin{bmatrix}2 & 4\\3 & k\end{bmatrix}$?

<u>Solution</u>

$$2k - 3 \cdot 4 = 12$$
$$2k = 12 + 12$$
$$\boxed{k = 12}$$

<u>Check</u> $\det \begin{bmatrix}2 & 4\\3 & 12\end{bmatrix} = 24 - 12 = 12$ ✓

14) Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

a) Find all 2 eigenvectors of A.

Solution (a)
Want $\begin{bmatrix} x \\ y \end{bmatrix}$ so that $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 3-2 & -2 \\ 1 & 0-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t$

$x - 2y = 0 \Rightarrow x - 2t = 0 \Rightarrow x = 2t$

All 2-eigenvectors of A are $\begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for all nonzero number t.

b) Find all -1 eigenvectors of A.

Solution (b)

$$\left[\begin{array}{cc|c} 3--1 & -2 & 0 \\ 1 & 0--1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$-\frac{1}{4}R_1 + R_2$ $\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right]$

$\frac{1}{2}y = 0 \Rightarrow \boxed{y = 0}$

$4x + 2y = 0 \Rightarrow 4x = 0 \Rightarrow \boxed{x = 0}$

The only solution to $A\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is not an eigenvector because it's the zero vector. So A has no -1 eigenvectors.