Midterm Practice for
Math 3333 Linear Algebra

Sources :

Lectures 1 to 8 and Lecture 9a only

Perform row reduce step by step

Compute the rank of a matrix

Solving a linear system

- no solution
- exactly one solution
- solution set with 1 parameter, 2 parameters or more

Transpose of a matrix

scalar multiplication

matrix addition

matrix multiplication

Showing that matrix multiplication does not commute

Finding matrices which commute with other matrices of the same size

Inverse of a matrix

Rearranging equations

Determining whether a matrix is invertible

- using rank
- using determinant
- knowing that if the matrix is not square then it's not invertible

Computing determinant

- using row reduce and upper triangular matrix
- using cofactor expansion

Computing eigenvectors

- Given a number λ, find eigenvectors or determine it doesn't exist

Eigenvalues and eigenvectors (see Q10, Q11)

- Given the solution set of a matrix equation, determine whether a vector is an eigenvector & whether a number is an eigenvalue

Geometric meaning of vector arithmetic in 2D

- scalar multiplication
- vector addition

Dimension and shape of a solution set (see Q3)

Performing sanity checks after computing a solution

Q 1

a. Walk me through the process of using
① augmented matrix and
② row reduce
to find all solutions to the linear system.

$$\begin{aligned} x + y + z &= 3 \\ x + y + 2z &= 4 \\ y + 2z &= 2 \end{aligned}$$

b. * Write down a matrix multiplication which you can perform to verify your solution.

* Perform the matrix multiplication.

Key

(Source Lecture 1b
Exercise 2)

a)

$$x + y + z = 3$$
$$x + y + 2z = 4$$
$$y + 2z = 2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & -1 & -2 & -4 \\ 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \mapsto R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\left.\begin{array}{r} x + y + z = 3 \\ -z = -1 \\ y + 2z = 2 \end{array}\right\}$$

$x + 0 + 1 = 3 \Rightarrow \boxed{x = 2}$

$\boxed{z = 1}$

$y + 2(1) = 2 \Rightarrow \boxed{y = 0}$

Solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

b. Write down a matrix multiplication which you can perform to verify your solution

$$* \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

* Perform the matrix multiplication.

$$\begin{bmatrix} 1.2 + 0 + 1 \\ 1.2 + 0 + 2.1 \\ 0 + 0 + 2.1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \checkmark$$

Q2

Let $M := \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix}$

a) There are many REF matrices equivalent to M. Walk me through the process of finding one REF matrix equivalent to M.

b) Use this REF matrix to tell me the rank of M.

a) Key (Source: Lecture 2a, 2b)

$M :=$

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \xrightarrow[R_2 \mapsto -2R_1 + R_2]{} \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \xrightarrow[R_3 \mapsto -R_1 + R_3]{} \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \mapsto \frac{1}{3} R_2]{} \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

one of the many REF matrices equivalent to M

b) The above REF matrix has two leading 1s, so $\operatorname{rank}(M) = 2$.

Q 3

* Describe all solutions to the (consistent) system

$$\begin{cases} a - 2b \qquad + d = 2 \\ \qquad\qquad c - 2d = 1 \end{cases}$$

* How many parameters are needed to describe all solutions?

* What is the dimension of this solution set?

* What is the shape of this solution set?

Key (Source: Lecture 2a)

Find all solutions to the system

$$\begin{cases} a - 2b \quad + d = 2 \\ \qquad\quad c - 2d = 1 \end{cases}$$

Think: $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$

The 2nd & 4th columns have no leading 1s.

Let $\boxed{b = t, \quad d = s}$

Then $c - 2d = 1 \Rightarrow c - 2s = 1 \Rightarrow \boxed{c = 1 + 2s}$

$a - 2b \quad + d = 2 \Rightarrow a - 2t + s = 2 \Rightarrow \boxed{a = 2 + 2t - s}$

The solutions are

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 + 2t - s \\ t \\ 1 + \quad 2s \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

for all numbers t and s.

* Two parameters are needed to describe the solutions
* The dimension is 2
* The shape is a plane

(Source: Lecture 9a)

Q 4

- If $M = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix}$,

  What is M^T ?

- What is the transpose of A^T ?

Key (Source: Lecture 3a)

- If $M = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix}$,

  What is M^T?

  $M^T = \boxed{\begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}}$

- What is the transpose of A^T?

  $(A^T)^T = \boxed{A}$

Q5

What is the size of the product

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \quad ?$$

What is the size of the product

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \quad ?$$

What is the size of the product

$$\begin{bmatrix} 1 & 2 & \pi & 5 \\ 3 & 0 & 0 & 2 \\ 0 & 9 & 1 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad ?$$

What is the size of the product

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & \pi & 5 \\ 3 & 0 & 0 & 2 \\ 0 & 9 & 1 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \quad ?$$

Key

Source: Lecture 4a Exercise 2

What is the size of the product

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \quad ?$$

$2 \times \cancel{3} \quad \cancel{3} \times 2$

Answer: $\boxed{2 \times 2}$

What is the size of the product

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \quad ?$$

$3 \times \cancel{2} \quad \cancel{2} \times 3$

Answer: $\boxed{3 \times 3}$

What is the size of the product

$$\begin{bmatrix} 1 & 2 & \pi & 5 \\ 3 & 0 & 0 & 2 \\ 0 & 9 & 1 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad ?$$

$5 \times \cancel{4} \quad \cancel{4} \times 1$

Answer: $\boxed{5 \times 1}$

What is the size of the product

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & \pi & 5 \\ 3 & 0 & 0 & 2 \\ 0 & 9 & 1 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \quad ?$$

$2 \times \cancel{5} \quad \cancel{5} \times 4$

Answer: $\boxed{2 \times 4}$

Q6

- Give me one 2×2 matrix which commutes with every 2×2 matrix

- Give me another 2×2 matrix which commutes with every 2×2 matrix.
  - How do you know it commutes with every 2×2 matrix?

- Is matrix multiplication commutative?

- Give me two (easy to remember) matrices which do not commute with each other.

Key

- Give me one 2x2 matrix which commutes with every 2x2 matrix

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the identity matrix commutes with every 2x2 matrix

- Give me another 2x2 matrix which commutes with every 2x2 matrix.

From quiz 1: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ works

$$\begin{cases} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} \end{cases}$$

- Is matrix multiplication commutative? No

- Give me two (easy to remember) matrices which do not commute with each other.

- If A has size 3×2 and B has size 2×3, AB and BA are defined but they have different sizes, so $AB \neq BA$

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 8 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 4 & 0 \end{bmatrix}$$

Ref: Matrix multiplication (Lecture 4a, 4b and Quiz 1)

Q7

- Suppose A, B, and C are 2×2 matrices and $\det(C) = 5$.

  Rewrite the matrix equation

$$AC - BC = 4CB$$

  as a formula for A.

- What should you do to verify your answer?

Key

Similar to Exercise 3 Lecture 5a

Suppose A, B, and C are 2×2 matrices and $\det(C) = 5$.

Rewrite the matrix equation

$$AC - BC = 4CB$$

as a formula for A.

Answer

- Since $\det(C) \neq 0$, C^{-1} exists.

$$AC - BC = 4CB$$

$$(A-B)C = 4CB$$

$$(A-B)CC^{-1} = 4CBC^{-1}$$

$$A - B = 4CBC^{-1}$$

$$A = 4CBC^{-1} + B$$

- Sanity check

$4CBC^{-1} + B$

$$AC - BC \stackrel{?}{=} 4CB$$

$$(4CBC^{-1}+B)C - BC$$

$$4CB\underbrace{C^{-1}C}_{Id} + \underbrace{BC - BC}_{0} = 4CB \checkmark$$

Q8

a) Let $M := \begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Without performing the $[M \mid Id] \rightsquigarrow [REF \mid \quad]$ algorithm,

determine whether M is invertible

b) The same question for

$$C := \begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Hint: For part (a), M is an upper triangular matrix,

so we can quickly compute $\det(M)$.

Do you remember how? (Look at the diagonal entries)

Key

Similar to Lecture 5b Exercise 9

a) $M := \begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is invertible

Why?
- $\det(M) = 2 \cdot 5 = 10 \neq 0$, and we know if the determinant of a matrix is nonzero then the inverse of the matrix exists
- Alternatively, the rank of M is 7 and its size is 7×7, so M is invertible according to the invertibility-and-rank theorem

b) $C := \begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$ is not invertible.

Why? C is equivalent to the 7×7 matrix $\begin{bmatrix} 1 & 0 & 0 & -1 & 9 & 8 & 0 \\ 0 & 1 & 7 & 3 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & 7 & 2 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which has determinant 0 and rank smaller than 7.

Q9

- Let $M := \begin{bmatrix} 1 & -1 & 7 & 0 & 1 \\ 0 & 2 & 6 & 0 & 1 \\ 7 & 5 & -6 & 2 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix}$

  Walk me through your process for computing $\det(M)$.

- Is M invertible?

Key

Source: Lecture 7a
Exercise 3

- A possible step-by-step process

For cofactor method, the most convenient choice is either the 4th column or 4th row.
Here, I choose the 4th Column

$$\det\begin{bmatrix} 1 & -1 & 7 & 0 & 1 \\ 0 & 2 & 6 & 0 & 1 \\ 7 & 5 & -6 & 2 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix} = \underbrace{0\cdot C_{14}}_{0} + \underbrace{0\cdot C_{24}}_{0} + 2\cdot C_{34} + \underbrace{0\cdot C_{44}}_{0} + \underbrace{0\cdot C_{54}}_{0}$$

$$= 2\cdot\underbrace{(-1)^{3+4}}_{-1} \det\begin{bmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 4 & 3 \end{bmatrix}$$

$R_4 \mapsto -R_1 + R_4$ does not change the determinant

$$= 2\cdot(-1) \det\begin{bmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \mapsto -3R_3 + R_4$ does not change the determinant

$$= -2\cdot \det\begin{bmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= -2 \qquad 1.2.-1.2$$

$$= \boxed{8}$$

- Yes, M is invertible because $\det(M) \neq 0$.

Con't below

(for alternative solution)

Key (Continued) Source: Lecture 7a Exercise 3

- A possible step-by-step process

For cofactor method, the most convenient choice is either the 4th column or 4th row.
Here, I choose the 4th row.

$$\det\begin{bmatrix}1&-1&7&0&1\\0&2&6&0&1\\7&5&-6&2&4\\0&0&-1&0&0\\1&-1&4&0&3\end{bmatrix} = a_{41}C_{41} + a_{42}C_{42} + a_{43}C_{43} + a_{44}C_{44} + a_{45}C_{45}$$

$$C_{41} = (-1)^{4+1}\det\begin{bmatrix}-1&7&0&1\\2&6&0&1\\5&-6&2&4\\-1&4&0&3\end{bmatrix}$$

the original matrix without the 4th row, 1st column

$$C_{42} = (-1)^{4+2}\det\begin{bmatrix}1&7&0&1\\0&6&0&1\\7&-6&2&4\\1&4&0&3\end{bmatrix}$$

the original matrix without the 4th row, 2nd column

I did not need to compute $C_{41}, C_{42}, C_{44}, C_{45}$ because $a_{41}=a_{42}=a_{44}=a_{45}=0$, but I wanted to show you examples for computing cofactors

$$C_{43} = \underbrace{(-1)^{4+3}}_{-1}\det\begin{bmatrix}1&-1&0&1\\0&2&0&1\\7&5&2&4\\1&-1&0&3\end{bmatrix}$$

the original matrix without the 4th row, 3rd column

Cofactor along the 3rd column

$$= -1\cdot\left[0 + 0 + 2\cdot(-1)^{3+3}\det\begin{bmatrix}1&-1&1\\0&2&1\\1&-1&3\end{bmatrix} + 0\right]$$

$$= -1\left(2\cdot\det\begin{bmatrix}1&-1&1\\0&2&1\\0&0&2\end{bmatrix}\right) = -1\cdot 2\cdot\underbrace{(1\cdot 2\cdot 2)}_{\text{diagonal entries of an upper triangular matrix}}$$

$$= -8$$

So, $\det\begin{bmatrix}1&-1&7&0&1\\0&2&6&0&1\\7&5&-6&2&4\\0&0&-1&0&0\\1&-1&4&0&3\end{bmatrix} = a_{41}C_{41} + a_{42}C_{42} + a_{43}C_{43} + a_{44}C_{44} + a_{45}C_{45}$

$$= 0\cdot C_{41} + 0\cdot C_{42} + (-1)\,C_{43} + 0\cdot C_{44} + 0\cdot C_{45}$$

$$= 0 + 0 + (-1)(-8) + 0 + 0$$

$$= \boxed{8}$$

- Yes, M is invertible because $\det(M) \neq 0$.

Q 10

- Find all eigenvectors of $A := \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ with eigenvalue 0

- How can you check this by hand?

- If $M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of M?

- If $M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean 0 is an eigenvalue of M?

- If $M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of M?

- Suppose $B \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

  Give me one of the eigenvalues of the matrix B.

Key (Source: lecture 7b Exercise 8)

Find all eigenvectors of $A := \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ with eigenvalue $\lambda = 0$

Set $A v = 0 v$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right]$$

$$R_1 \mapsto \tfrac{1}{2} R_1 \quad \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right]$$

$$R_2 \mapsto R_1 + R_2 \quad \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t$

$$2x - y = 0 \Rightarrow 2x - t = 0 \Rightarrow 2x = t \Rightarrow x = \tfrac{1}{2} t$$

The eigenvectors of A with eigenvalue $\lambda = 0$ are of the form

$$\boxed{\begin{pmatrix} \frac{1}{2} t \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \text{ for nonzero } t.}$$

- Check Compute $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} t \\ t \end{bmatrix}$. Is it equal to $0\begin{bmatrix} \frac{1}{2} t \\ t \end{bmatrix}$?

- If $M\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of M?

  Yes. This means $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue 0

- If $M\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean 0 is an eigenvalue of M?

  Yes.

- If $M\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, does it mean $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of M?

  No. Eigenvector cannot be a zero matrix.

- Suppose $B\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$. Give me one the eigenvalues of the matrix B.

  - $\lambda = 2$ is an eigenvalue of B because

  $$B\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Q11

Suppose M is a 4×4 matrix.

Suppose $\left(M - 5\,\mathrm{Id}\right)\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has a unique solution.
(exactly one)

- Is it enough information to determine whether 5 is an eigenvalue of M?
  (If so, state whether 5 is an eigenvalue.)
  Explain.

- Is it enough information to determine whether -5 is an eigenvalue of M?
  (If so, state whether -5 is an eigenvalue.)
  Explain.

Suppose $\left(M - 5\,\mathrm{Id}\right)\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has infinitely many solutions.

- Is it enough information to determine whether 5 is an eigenvalue of M?
  (If so, state whether 5 is an eigenvalue.)
  Explain.

- Is it enough information to determine whether -5 is an eigenvalue of M?
  (If so, state whether -5 is an eigenvalue.)
  Explain.

Suppose M is a 4×4 matrix.

Suppose $(M - 5\,\mathrm{Id})\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ has a unique solution.
(exactly one)

- Is it enough information to determine whether 5 is an eigenvalue of M? (If so, state whether 5 is an eigenvalue.) yes.

  This means the only $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ satisfying $M\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = 5\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ is $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$,

  so 5 is <u>not</u> an eigenvalue of M.

- Is it enough information to determine whether -5 is an eigenvalue of M? (If so, state whether -5 is an eigenvalue.)

  Not enough information about -5

Suppose $(M - 5\,\mathrm{Id})\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ has infinitely many solutions.

- Is it enough information to determine whether 5 is an eigenvalue of M? (If so, state whether 5 is an eigenvalue.) yes.

  This means there are non-zero vectors $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ satisfying $M\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = 5\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$

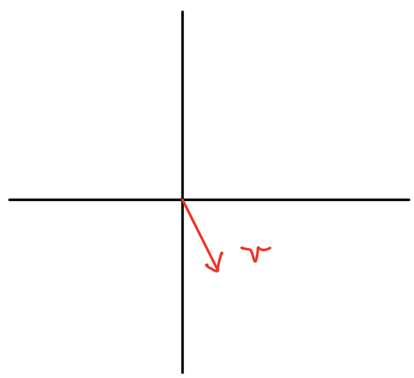
  so 5 is an eigenvalue of M.

- Is it enough information to determine whether -5 is an eigenvalue of M? (If so, state whether -5 is an eigenvalue.)

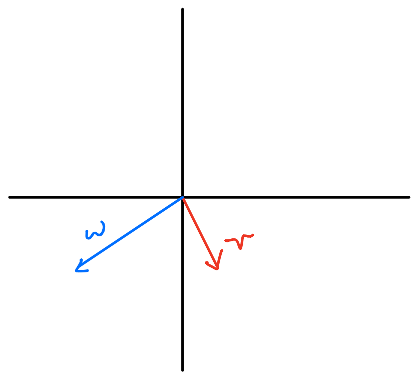
  Not enough information about -5

Q 12

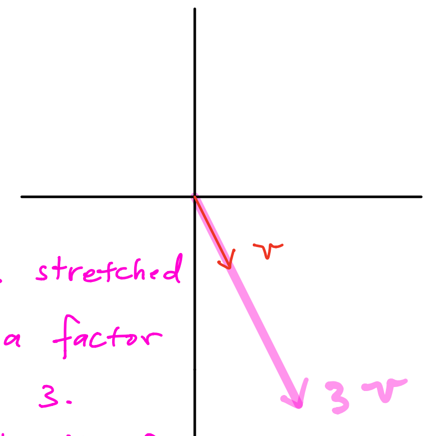
* Use the annotate function to sketch $3v$ and $\frac{1}{2}v$
  if v is shown below



* On your paper, sketch $v + w$,
  where v and w are shown below

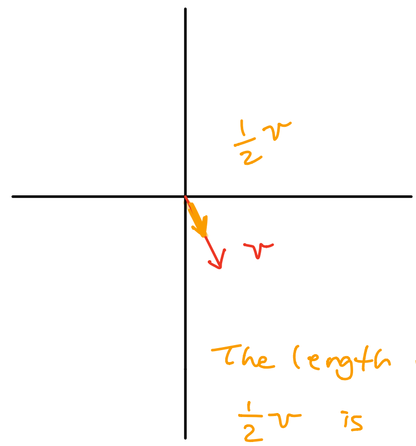


* Use the annotate function to sketch $3v$, if v is shown below



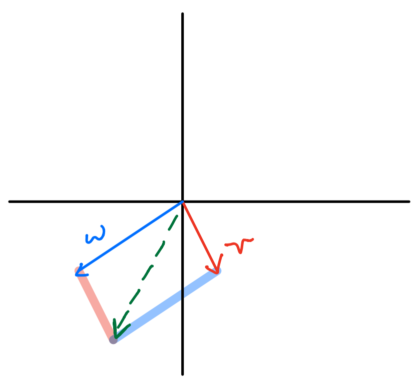
v is stretched by a factor of 3.

The length of $3v$ is three times $|v|$.



The length of $\frac{1}{2}v$ is half of $|v|$

* On your paper, sketch $v + w$, where v and w are shown below



$v + w$ are drawn in dashed line