Lecture 9b

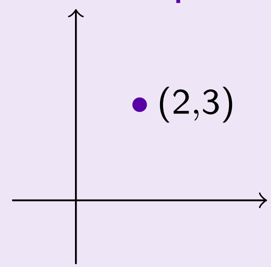
Vector Geometry with matrices

Review: Interpreting 2-vectors geometrically

We can visualize 2-vectors in the plane in two ways.

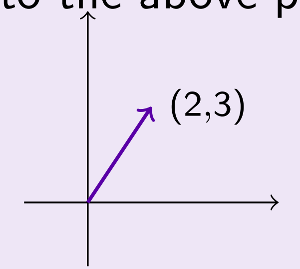
- Interpret the entries as coordinates of a point.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \xrightarrow{\text{...becomes...}}$$



- Draw an arrow from the origin to the above point.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \xrightarrow{\text{...becomes...}}$$



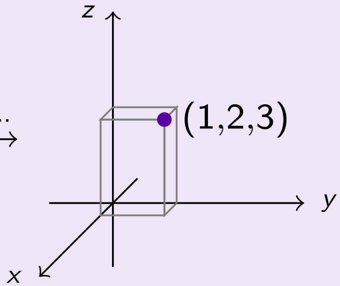
Confusingly, this arrow is often called a (geometric) vector.

Review: Interpreting 3-vectors geometrically

We can do the same thing for 3-vectors.

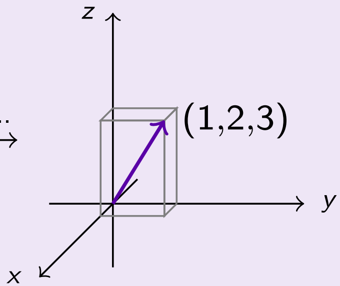
- Interpret the entries as coordinates of a point.

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \xrightarrow{\text{...becomes...}}$$



- Draw an arrow from the origin to the above point.

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \xrightarrow{\text{...becomes...}}$$

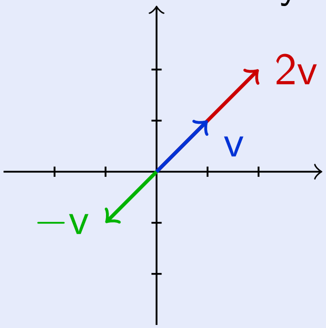


There is no standard for which variable corresponds to each axis.

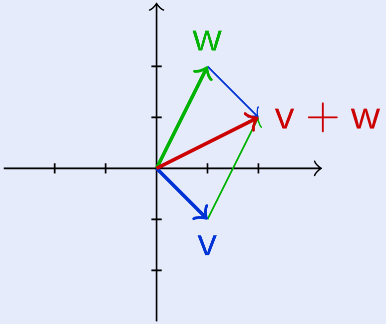
Review: scalar multipl and vector addition → geometry

From algebra to geometry

- Multiplying v by a scalar c stretches c by a factor of c.



- Adding v and w gives the new vector obtained by sliding the tail of one vector to the tip of the other.



What about matrices?

Linear transformations: the idea

The geometric analog of a matrix isn't an object, but a transformation that acts on vectors (or points).

This idea will be very useful even outside of geometric pictures.

Examples of linear transformations

- Rotations
- Reflections
- Projections
- Many more!

The linear transformation of a matrix (Definition)

Given a matrix A, the **linear transformation of A** is the function

$$T_A(v) = \underbrace{Av}$$

T_A takes in a vector v

In order for the function T_A to make sense, the product Av must be defined

- The input v of T_A is a vector whose height must be width(A).
- The output of T_A is a vector, Av, whose height is height(A).

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 5 \\ 9 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$3 \times 4 \quad 4 \times 1 \qquad 3 \times 1$

Example

The linear transformation of A $:= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is defined by

$$\begin{aligned} T_{\mathrm{A}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix} \text{ for any } x, y \end{aligned}$$

Note that this function linear transformation T_A takes in a 2-vector and returns a 2-vector.

Exercise 2

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(a) Evaluate $T_A\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)$, $T_A\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$, and $T_A\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$.

(b) Find a vector v such that $T_A(v) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2a By def, $T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$T_A\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3-4 \\ 9-8 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note: Any linear transformation sends a zero vector to a zero vector (of appropriate height)

Exercise 2

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

ⓑ Find a vector v such that $T_A(v) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2b

We want $T_A(v) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find v.

Side note: In order for $T_A(v)$ to make sense, the product $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} v$ must be defined. So the height of v should be width$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 2$.

Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$. We'll search for x and y.

$$T_A(v) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{(by definition)}$$

$$= \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix}$$

Set $T_A(v) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then $\begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\left.\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= 1 \end{aligned}\right\} \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 1 \end{array}\right]$$

$$R_2 \mapsto -3R_1 + R_2 \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array}\right]$$

$$R_2 \mapsto -\tfrac{1}{2}R_2 \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \left.\begin{aligned} x+2y &= 1 \\ \boxed{y=1} \end{aligned}\right\} \Rightarrow x + 2 = 1 \Rightarrow \boxed{x = -1}$$

$$\therefore \; v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Sanity check

$$T_A\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2 \\ -3+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

ⓑ Find a vector v such that $T_A(v) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:)

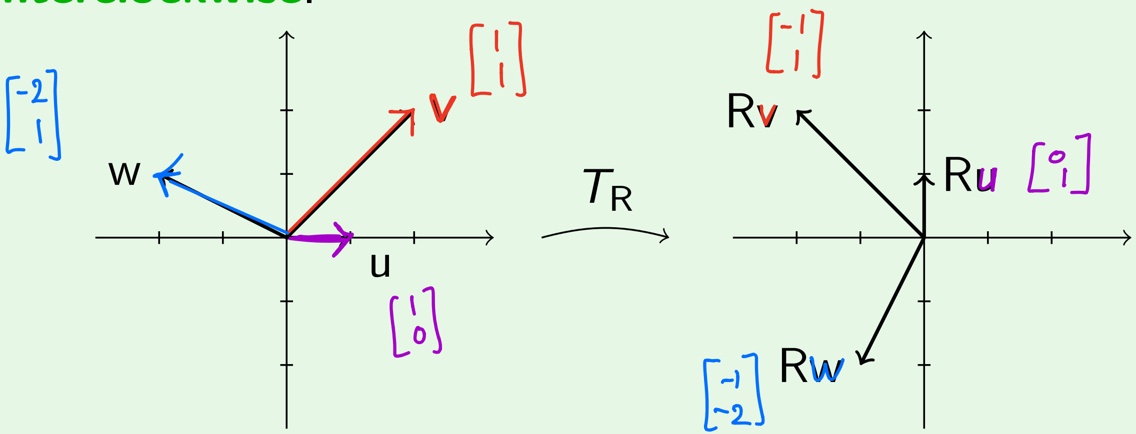
Many natural geometric transformations are linear.

Example

Let $\mathrm{R} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then the linear transformation is

$$T_{\mathrm{R}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

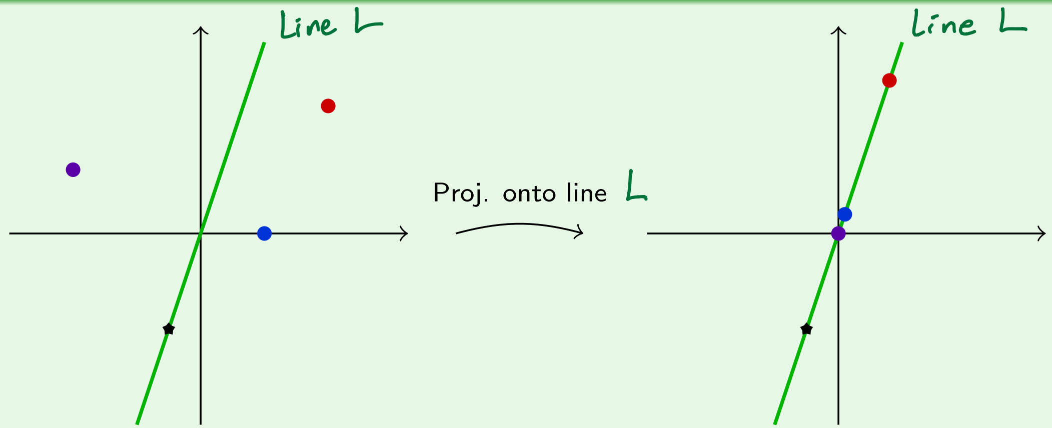
Geometrically, this takes in a vector and **rotates it** 90° **counterclockwise**.



Projections

The projection of a point v onto a line (or a plane) is the closest point to v in that line (or a plane).

Example: Projection onto a line



Projections are useful in many applications because they give us the closest approximation of v by points on a set.

A formula for projection

If L is the line through the origin and the point w, then

$$\text{the projection of v onto } L \text{ is } \frac{\mathrm{v}\cdot\mathrm{w}}{\mathrm{w}\cdot\mathrm{w}}\mathrm{w}$$

we can write this as a matrix multiplication!

Exercise 3

Let L be the line in the plane $\mathbb{R}^2$ through $(0,0)$ and $(1,3)$.

1. Find a formula for the projection of $\mathrm{v} = \underbrace{\begin{bmatrix} a & b \end{bmatrix}^\top}_{=\begin{bmatrix} a \\ b \end{bmatrix}}$ onto L.
2. Find a matrix M so that, for all points v,

$$T_M(\mathrm{v}) \text{ is the projection of v onto } L$$

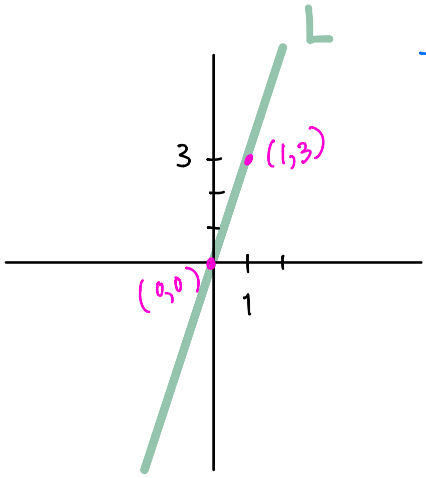
3 1

The projection of $v = \begin{bmatrix} a \\ b \end{bmatrix}$ onto L is $\frac{v \cdot w}{w \cdot w} w$

Here, $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$v \cdot w = a + 3b$

$w \cdot w = 1 \cdot 1 + 3 3 = 10$



The projection of $v = \begin{bmatrix} a \\ b \end{bmatrix}$

onto L is $\dfrac{v \cdot w}{w \cdot w}\, w = \underbrace{\dfrac{a+3b}{10}}_{\text{a number}} \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\text{a vector}}$

(scalar multiplication) $= \begin{bmatrix} \dfrac{a+3b}{10} \cdot 1 \\[2ex] \dfrac{a+3b}{10} \cdot 3 \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{a}{10} + \dfrac{3}{10} b \\[2ex] \dfrac{3}{10} a + \dfrac{9}{10} b \end{bmatrix}$$

The end of part 1

Exercise 3

Let L be the line in the plane $\mathbb{R}^2$ through $(0,0)$ and $(1,3)$.

② Find a matrix M so that, for all points v,

$$T_M(\mathrm{v}) \text{ is the projection of v onto } L$$

Want M so that $T_M(v)$ is a projection of $v = \begin{bmatrix} a \\ b \end{bmatrix}$ onto the line L

- What size should M be?
  - We want T_M to take in a $\underbrace{\text{2-vector}}$ (a point in $\mathbb{R}^2$) and to output another $\underbrace{\text{2-vector}}$ (a point in $\mathbb{R}^2$).
  - So the size of M should be $\underbrace{2}_{\text{height of } M} \times \underbrace{2}_{\text{width of } M}$

- Want $T_M\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} .1a + .3b \\ .3a + .9b \end{bmatrix}$ from part (1)

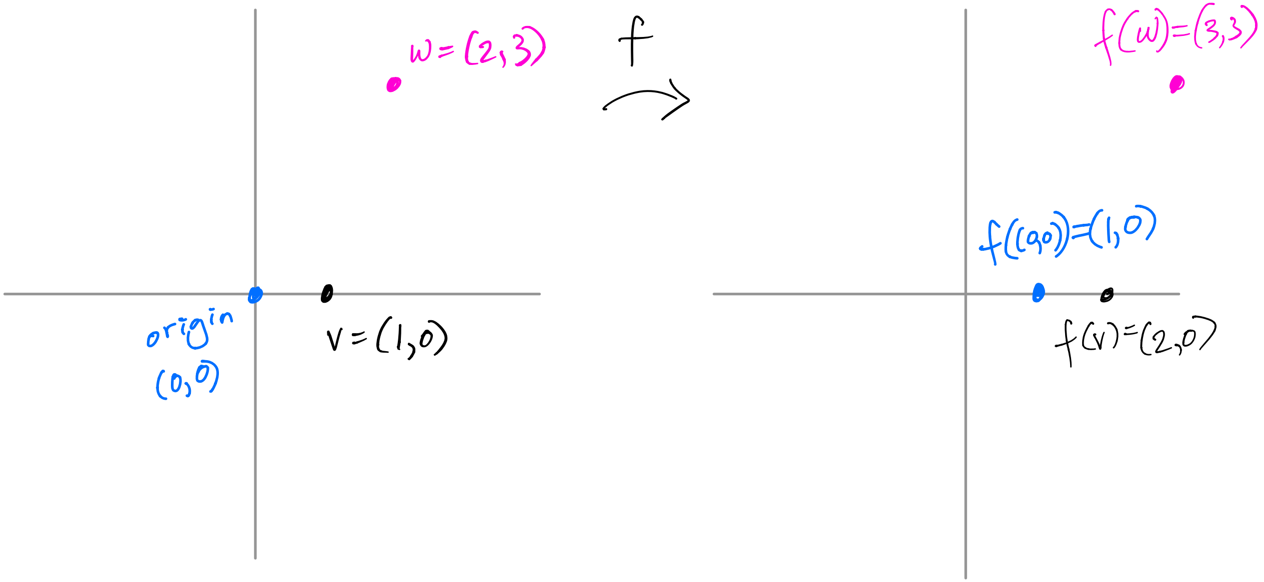
$$M \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Thus, $M = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$

$$\boxed{\text{This projection can be written as the linear transformation of the matrix } M = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}}$$

Not every "natural" transformation on vectors is linear!

Consider the transformation: translation to the right by 1



Not every "natural" transformation on vectors is linear!

$\boxed{\text{Exercise 4}}$

(a) Let f be the function which translates $\begin{bmatrix} x \\ y \end{bmatrix}$ by 1.
Give a formula for f.

(b) Show that f is not a linear transformation T_M for any matrix M.
(Hint: Where does f send the origin?)

$\boxed{\text{Exercise } 4}$

(a) Let f be the function which translates $\begin{bmatrix} x \\ y \end{bmatrix}$ by 1.
Give a formula for f.

Answer: $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y \end{bmatrix}$

(b) Show that f is not a linear transformation T_M for any matrix M.
(Hint: Where does f send the origin?)

side note: This is a trick which we can often use to show that a function is not a linear transformation

Answer

Suppose $T_M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y \end{bmatrix}$ for some $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y \end{bmatrix}$.

Plugging in $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we get $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} a.0 + b.0 \\ c.0 + d.0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, we have $0 = 1$.

This $\boxed{\text{contradicts}}$ the fact that the number 0 is not equal to the number 1.

Thus, the function f is not a linear transformation T_M for any matrix M.

~ the end of the argument ~

Side note: One of the key ideas is contradiction.

$\boxed{\text{Exercise 4}}$

Use this page as a template to answer similar questions later

(a) Let f be the function which translates $\begin{bmatrix} x \\ y \end{bmatrix}$ by 1. Give a formula for f.

Answer: $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y \end{bmatrix}$

(b) Show that f is not a linear transformation T_M for any matrix M.

Answer

Suppose $T_M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y \end{bmatrix}$ for some $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y \end{bmatrix}$.

Plugging in $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we get $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} a.0+b.0 \\ c.0+d.0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, we have $0 = 1$.

This contradicts the fact that the number 0 is not equal to the number 1.

Thus, the function f is not a linear transformation T_M for any matrix M.

As we saw, not every transformation on vectors is linear.

Next time

- What special properties do linear transformations have?
- How can you tell if a transformation is linear?
- If so, how can you write it as T_A for some matrix A?