

## Lecture 9a

# Vector Geometry

So far, we've focused on algebraic computations.  
From now on, we shift focus to concepts and intuition.

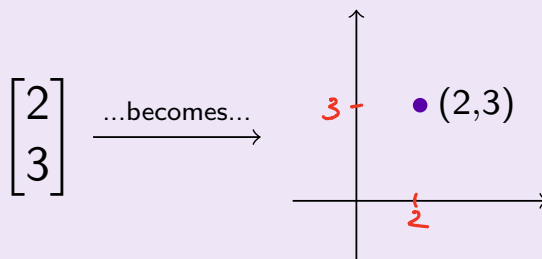
## Lecture 9 and 10

Visualizing vectors of height 2 and 3 geometrically, and translating ideas from the class so far into geometry.

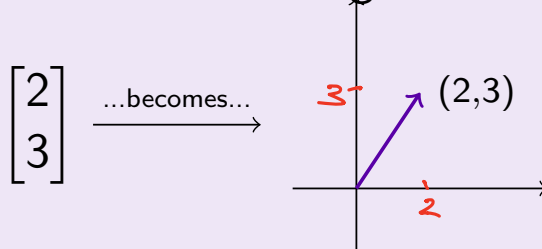
## Interpreting 2-vectors geometrically

We can visualize 2-vectors in the plane in two ways.

- We can interpret the entries as **coordinates of a point**.



- We can draw an **arrow** from the origin to the above point.

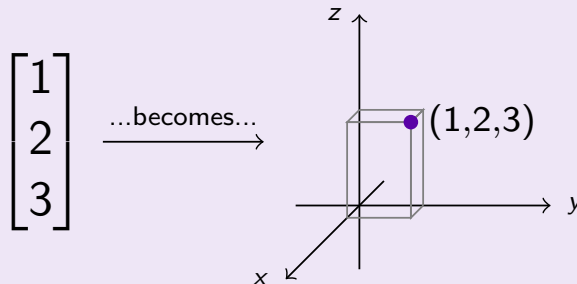


Confusingly, this arrow is often called a **(geometric) vector**.

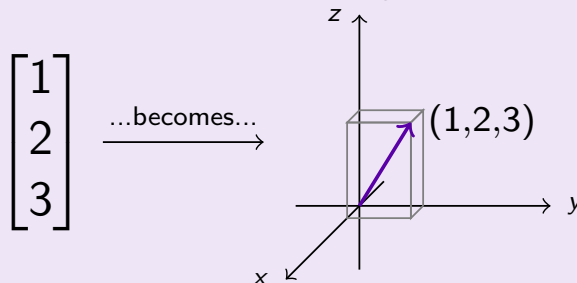
## Interpreting 3-vectors geometrically

We can do the same thing for 3-vectors.

- We can interpret the entries as **coordinates of a point**.



- We can draw an **arrow** from the origin to the above point.



There is no standard for which variable corresponds to each axis.

## Interpreting bigger vectors geometrically?

As 3D people, you can't visualize 4-dimensional space or higher directly. So we can't visualize larger vectors geometrically.

That said, we can run the intuition the other way, and use our algebraic knowledge of vectors to understand higher dimensional space.

### Exercise 1(a)

Draw all solutions to the following system of linear equations.

$$x + y = 3$$

$$x - y = 1$$

$$y = 1$$

### Exercise 1(b)

Draw all solutions to the following system of linear equations.

$$x + y = 3$$

$$x - y = 1$$

$$x + y = 1$$

### Exercise 1(c)

Draw all solutions to the following system of linear equations.

$$x + y = 3$$

$$-x - y = -3$$

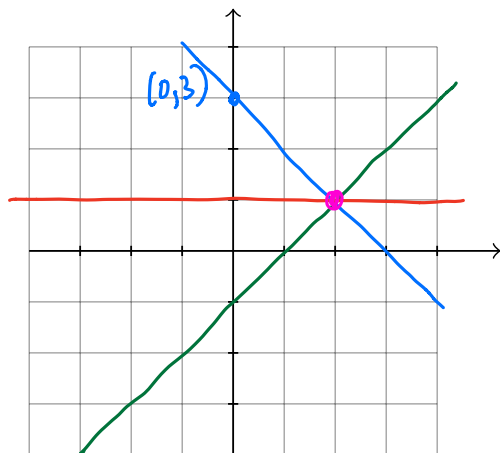
$$2x + 2y = 6$$

1a

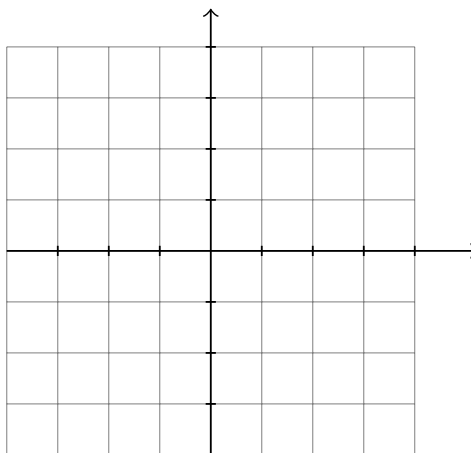
$$\begin{aligned} x + y &= 3 \quad (\Leftrightarrow y = -x + 3) \\ x - y &= 1 \quad (\Leftrightarrow y = x - 1) \\ y &= 1 \end{aligned}$$

slope is -1

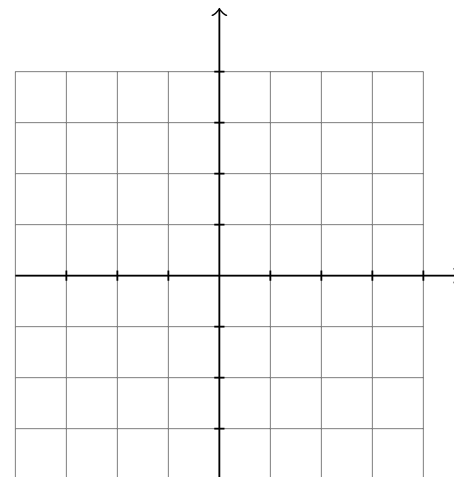
slope is 1



$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \\ x + y &= 1 \end{aligned}$$



$$\begin{aligned} x + y &= 3 \\ -x - y &= -3 \\ 2x + 2y &= 6 \end{aligned}$$



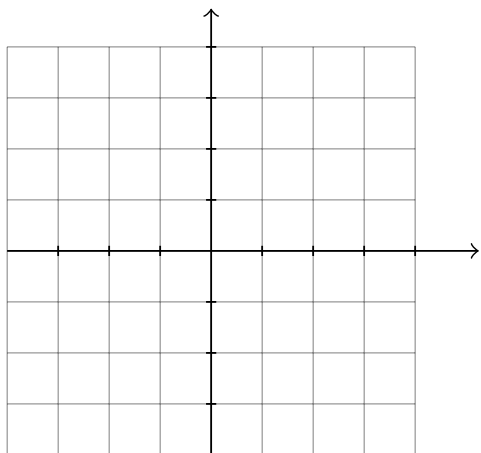
This system has  
exactly one  
solution,  $(x, y) = (2, 1)$

16

$$x + y = 3$$

$$x - y = 1$$

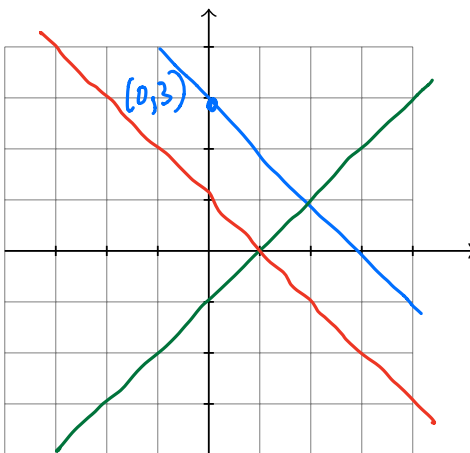
$$y = 1$$



$$x + y = 3 \Leftrightarrow y = -x + 3 \quad \text{slope is } -1$$

$$x - y = 1 \Leftrightarrow y = x - 1 \quad \text{slope is } 1$$

$$x + y = 1 \Leftrightarrow y = -x + 1 \quad \text{slope is } -1$$

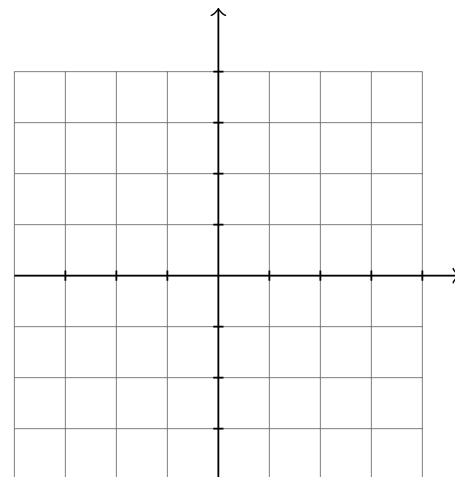


No points are  
solutions to this  
system

$$x + y = 3$$

$$-x - y = -3$$

$$2x + 2y = 6$$

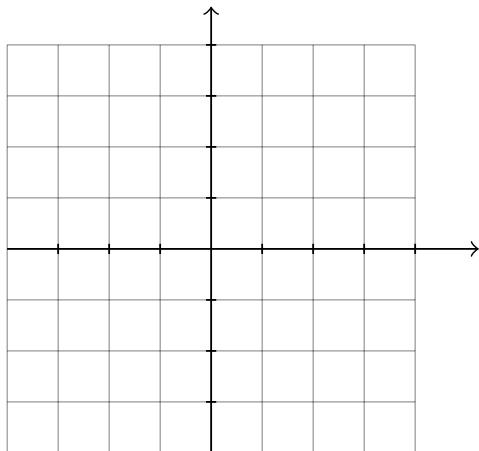




$$x + y = 3$$

$$x - y = 1$$

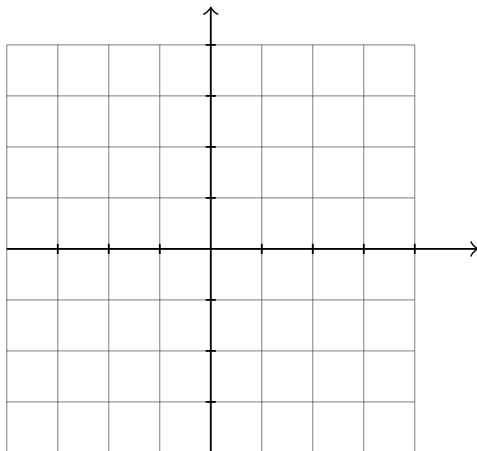
$$y = 1$$



$$x + y = 3$$

$$x - y = 1$$

$$x + y = 1$$

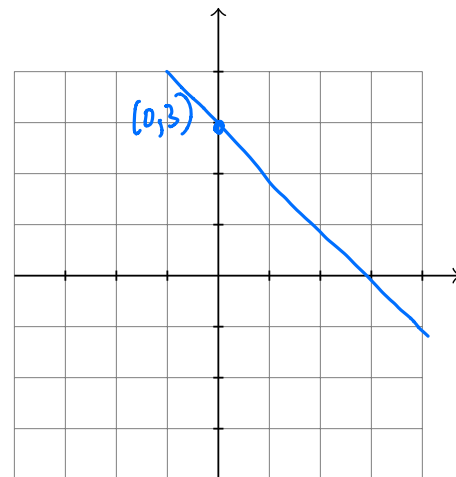


slope is -1

$$y = -x + 3 \Leftrightarrow x + y = 3$$

$$y = -x + 3 \Leftrightarrow -x - y = -3$$

$$y = -x + 3 \Leftrightarrow 2x + 2y = 6$$



The solutions are  
points  $(x, y) = (t, -t + 3)$   
where  $t$  is a number

Now is a good time to introduce the idea of **sets**.

## Sets

A **set** is a collection of objects, called **elements**...which are typically mathematical objects like numbers, vectors, or points.

## Examples of sets

- The set of all cats in Oklahoma — a finite set
- The set of all even integers — an infinite set
- The set of solutions to a system of linear equations
- The set of eigenvalues of a matrix
- The set of all 2-vectors — an infinite set

can have 0 elts

can have exactly one element

can have infinitely many elements

→ always a finite set  
(an  $n \times n$  matrix can have at most  $n$  eigenvalues)

## Defining sets

Sets can be defined by listing their elements...

$$\{\text{cat } 1, \text{cat } 2, \dots, \text{cat } d\}, \quad \{-1, 2, 7\}, \quad \{\dots, -4, -2, 0, 2, 4, 6, 8, \dots\}$$

...parametrizing their elements...

$$\left\{ \begin{bmatrix} 1 - 2t \\ t \end{bmatrix} \text{ for all numbers } t \right\}, \quad \{2k \text{ for all integers } k\}$$

...or characterized by a property.

$$\text{the set of solutions to } x + 2y + 3y = 6, \quad \text{the set of integers divisible by 2}$$

## No elements?!

A set with no elements is called the **empty** set.

## Graphing sets of vectors

We can visualize a set of 2-vectors or 3-vectors by drawing the corresponding points in the plane or space, respectively.

### Example

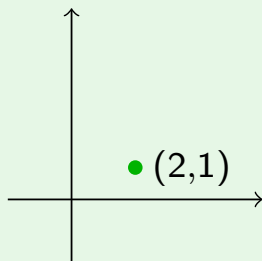
system of  
linear equations  
(SLE)

$$x + y = 3$$

$$x - y = 1$$

$$y = 1$$

Solution set

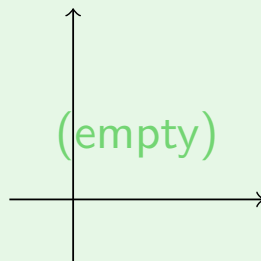


$$x + y = 3$$

$$x - y = 1$$

$$x + y = 1$$

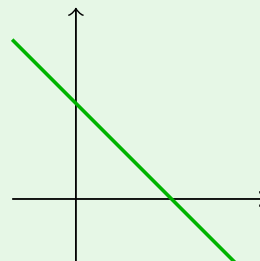
(empty)



$$x + y = 3$$

$$-x - y = -3$$

$$2x + 2y = 6$$



What kinds of shapes can we get as the set of solutions to an SLE?  
Let's start with one linear equation.

## The shape of solutions to a linear equation

- The set of solutions to a linear equation in 2-variables

$$ax + by = c \quad , \text{ e.g. } x - y = 0$$

or  $2x + 5y = -\pi$

is a line in the plane.

- The set of solutions to a linear equation in 3-variables

$$ax + by + cz = d \quad , \text{ e.g. } x + y + z = 1$$

or  $x - 5y - z = \frac{1}{7}$

is a plane in space.

We don't have the vocabulary for cases in higher dimension.

## One exception

Technically,  $0 = 1$  is a linear equation whose solutions are empty.

## The shape of solutions to a system of linear equations

The solution set to a system of linear equations has a fixed shape that depends on the number of solutions and parameters.

# of solutions	Shape
No solutions	No points (empty)
One solution	One point
One parameter	A line
Two parameters	A plane
$m$ parameters	?

$$\left[ \begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Let  $y = t$   
 $x - y = 0 \Rightarrow x = t$   
 $z = 2$

The solution set is

$\left\{ \begin{bmatrix} t \\ t \\ 2 \end{bmatrix} \text{ for a number } t \right\}$   
 or  $\left\{ t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \text{ for a number } t \right\}$

→ dimension of the solution set is 1

For this reason, the number of parameters needed is called the **dimension** of the solution set.

## The shape of solutions to a system of linear equations

The solution set to a system of linear equations has a fixed shape that depends on the number of solutions and parameters.

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Let  $y=t$   
 $z=s$

$$x - y + z = 1 \Rightarrow$$

$$x - t + s = 1 \Rightarrow$$

$$x = 1 + t - s$$

The solution set is

$$\left\{ \begin{bmatrix} 1+t-s \\ t \\ s \end{bmatrix} \text{ for all numbers } t, s \right\} =$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ for all numbers } t, s \right\}$$

# of solutions	Shape
No solutions	No points (empty)
One solution	One point
One parameter	A line
<u>Two parameters</u>	<u>A plane</u>
$m$ parameters	?

→ dimension of the solution set is 0

→ dimension of the solution set is 1

→ dimension of the solution set is 2

For this reason, the number of parameters needed is called the **dimension** of the solution set.

We will often denote a set by a letter, for example:

Let  $L$  be the set of solutions to  $2x - 3y = 3$ .

$$\begin{aligned} \text{Then } L &= \left\{ \begin{bmatrix} \frac{3}{2} + \frac{3}{2}t \\ t \end{bmatrix} \text{ for all } t \right\} \\ &= \left\{ \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \text{ for all } t \right\} \end{aligned}$$

A few sets come up so often that they have standard names.

### Standardized set names

- $\mathbb{R}$  is the set of (real) numbers.
- $\mathbb{R}^2$  is the set of 2-vectors, which we can visualize as the plane.
- $\mathbb{R}^3$  is the set of 3-vectors, which we can visualize as space.
- For any  $n$ ,  $\mathbb{R}^n$  is the set of  $n$ -vectors.
- For any  $m, n$ ,  $\mathbb{R}^{m \times n}$  is the set of  $m \times n$ -matrices.



We can extend our dictionary between algebra and geometry.

## *n*-dimensional geometry

Many of the formulas from 2D and 3D geometry extend to vectors of all sizes.

### Geometry → algebra

- The **length** of a vector  $v := [v_1 \ v_2 \ \cdots \ v_n]^T$  is

$$|v| := \sqrt{v_1^2 + v_2^2 + \cdots v_n^2} = \sqrt{v \cdot v}$$

- The **angle** between two vectors  $v$  and  $w$  can be defined by

$$\cos(\text{the angle between } v \text{ and } w) = \frac{v \cdot w}{|v||w|}$$

## Example

The **length** of the four-dimensional vector  $v := \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$  is

$$|v| = \sqrt{v \cdot v} =$$

Compute the **angle** between  $v$  and  $v$ .

Answer:

$$\cos(\boxed{\phantom{00}}) = \frac{v \cdot v}{|v||v|}$$

$$=$$

## Example

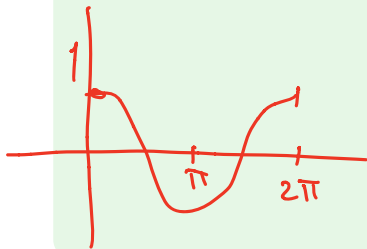
The **length** of the four-dimensional vector  $v := \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$  is

$$|v| = \sqrt{v \cdot v} = \sqrt{1^2 + 2^2 + 1^2 + 1^2} = \sqrt{7}$$

Compute the **angle** between  $v$  and  $v$ .

Answer:

$$\begin{aligned} \cos(\boxed{\phantom{0}}) &= \frac{v \cdot v}{|v||v|} \\ &= \frac{1^2 + 2^2 + 1^2 + 1^2}{\sqrt{7}\sqrt{7}} = \frac{7}{7} = 1 \end{aligned}$$



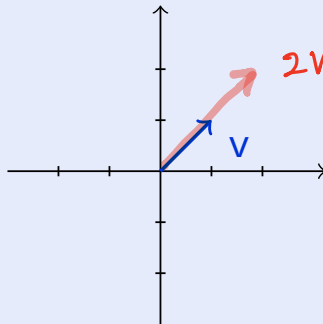
$\cos(\boxed{?}) = 1$   
 $\therefore$  The angle between  $v$  and  $v$  is  $\boxed{0}$ .

Dually, we can try to move algebraic ideas into geometry.

## From algebra to geometry

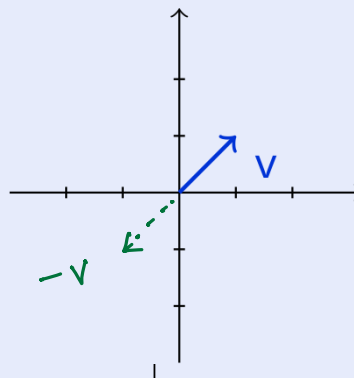
- Multiplying  $v$  by a scalar  $c$  stretches  $c$  by a factor of  $c$ .

If  $c := 2$   
and  $v := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



- Multiplying  $v$  by a scalar  $c$  stretches  $c$  by a factor of  $c$ .

If  $c := -1$   
 $v := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

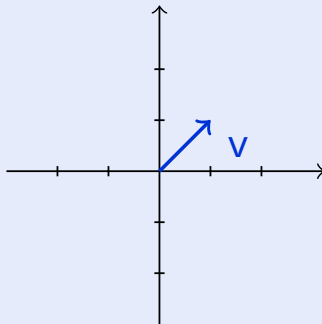


Dually, we can try to move algebraic ideas into geometry.

## From algebra to geometry

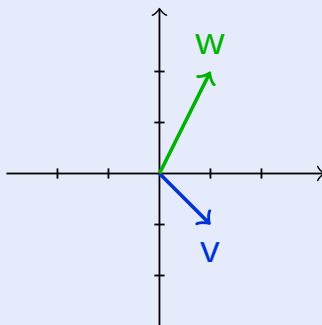
- Multiplying  $v$  by a scalar  $c$  **stretches**  $c$  by a factor of  $c$ .

$$cV = ?$$



- Adding  $v$  and  $w$  gives the new vector obtained by sliding the **tail** of one vector to the **tip** of the other.

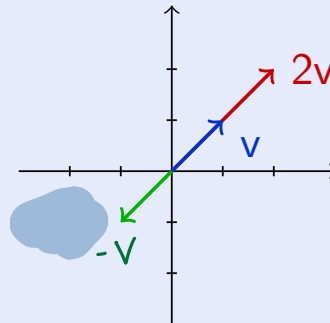
$$v + w = ?$$



Dually, we can try to move algebraic ideas into geometry.

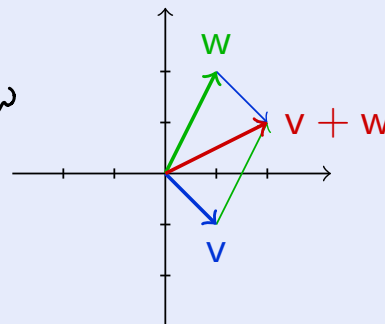
## From algebra to geometry

- Multiplying  $v$  by a scalar  $c$  **stretches**  $c$  by a factor of  $c$ .



- Adding  $v$  and  $w$  gives the new vector obtained by sliding the **tail** of one vector to the **tip** of the other.

if  $v$  is not  
a scalar  
multiple of  $w$



- ▶ Today: Vector geometry
- ▶ Next time: Matrix algebra to geometry