Extra Problems week 9 Wed Oct 21, 2020

Lec 9a

Q What shape is this linear equation?
(in 2 variables)

$$2x + 3y = 4$$

Does it contain the origin?

Ans
- A line in the plane
- No

Q What shape is this linear equation?
(in 3 variables)

$$2x + 3y + 4z = 5$$

Does it contain the origin?

Ans
- a plane in 3D space
- No

Q Does this shape contain the origin?

a. $2x + 3y + 4z + 5w = 6$

b. $2x + 3y + 4z + 5w = 0$

Ans a. No
b. yes

Lec 9b

Let $A := \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$.

Q: Find a vector v s.t $T_A(v) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Solution:

Row reduce the augmented matrix :

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{array}\right] \xrightarrow[\text{swap } R_1, R_3]{} \left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[R_1 \mapsto \frac{1}{2}R_1]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto -R_1 + R_3]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right] \xrightarrow[R_3 \mapsto R_2 + R_3]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

in REF

$$\boxed{v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Lec 10 a

Q Domain of T_A is $\mathbb{R}^2$ because #cols of A is 2

Q Target of T_A is $\mathbb{R}^3$ because # rows of A is 3

Q "T_A preserves addition" means ...

$$T_A(v + w) = T_A(v) + T_A(w)$$

Q "T_A preserves scalar multiplication" means ...

$T_A(rv) = r\,T_A(v)$ for all r in $\mathbb{R}$ and v in the domain of T_A.

Q "T_A preserves linear combination" means ...

Q Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation.

(Warning: The matrix of linear transformation is not provided!)

Suppose we know

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}2\\-1\\2\end{bmatrix}.$$

Find $T\left(\begin{bmatrix}3\\2\end{bmatrix}\right)$

Solution:

Step 1: Write $\begin{bmatrix}3\\2\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$.

Step 2: Use the property "T preserves linear combination"

Step 1: Want $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$

$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 - c_2 &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

$$\left.\begin{aligned} c_1 + c_2 &= 3 \\ \boxed{c_2 = \tfrac{1}{2}} \end{aligned}\right\} \Rightarrow c_1 + \frac{1}{2} = 3 \Rightarrow \boxed{c_1 = 4}$$

$$\boxed{c_1 = \frac{5}{2}}$$

$$\therefore \quad \frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

Step 2:

$$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = T\left(\frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$= \frac{5}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$= \frac{5}{2}\begin{bmatrix}0\\1\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\-1\\2\end{bmatrix}$$

$$= \begin{bmatrix}0+\frac{1}{2}\cdot 2\\ \frac{5}{2}-\frac{1}{2}\\ 5+1\end{bmatrix} = \begin{bmatrix}1\\2\\6\end{bmatrix}.$$

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lec 10b

Q What are the standard basis vectors in $\mathbb{R}^5$?

$$e_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \quad e_2 = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \quad e_3 = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \quad e_4 = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \quad e_5 = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

Thm If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $f = T_A$, where

$$A = \underbrace{\begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix}}_{m \times n}$$

Q: Suppose $f: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation.

Suppose we know $f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\2\end{bmatrix}$

and $f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\\0\end{bmatrix}$

Find the matrix A where $f = T_A$.

<u>Answer</u>

$$A = \begin{bmatrix} f(e_1) & f(e_2) \end{bmatrix}$$

$$= \begin{bmatrix} f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

<u>Sanity check</u>

- A has size 3×2

  $f: \mathbb{R}^2 \to \mathbb{R}^3$ good ✓

- Check $T_A\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \stackrel{?}{=} f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix} \checkmark$$

Review from earlier

If a SLE in variables x_1, x_2, x_3, x_4, x_5

is equivalent to the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 7 \end{array}\right], \text{ that is, } \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

write the solution set (possibly empty):

Ans Note $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 7 \end{array}\right]$ is already in REF.

We see that col 2 and col 5 have no leading 1s,

so let $x_2 = s$, $x_5 = t$.

We have $x_4 = 7$, $x_3 = 6$

and $x_1 + 2x_2 + 3x_5 = 5$

$\Rightarrow x_1 + 2s + 3t = 5$

$\Rightarrow x_1 = 5 - 2s - 3t$

So the solution set consists of vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 - 2s - 3t \\ s \\ 6 \\ 7 \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 7 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for s, t in $\mathbb{R}$