Lecture 8b

Characteristic Polynomials, second part

Review: Finding eigenvalues and eigenvectors

Recall: Eigenvectors and eigenvalues of a matrix

An **eigenvector** of an $n \times n$ matrix A is a *non-zero vector* v *with*

$$\mathrm{A}v = \lambda v$$

for some number λ, called the eigenvalue of the eigenvector v.

Note: it's possible that λ is 0.

Recall: Finding eigenvectors with a given eigenvalue

The λ-eigenvectors of A are the non-zero solutions to the matrix equation

$$(\mathrm{A} - \lambda \,\mathrm{Id})\mathrm{v} = \vec{0}$$

Recall: Finding eigenvalues of A

The eigenvalues of A are the roots of the char. poly. $p_{\mathrm{A}}(x) = \det(x\,\mathrm{Id} - \mathrm{A})$ of A.

Consequences of characteristic polynomials

Fact: A degree n polynomial has at most n distinct roots.
This means ...

Fact A (The number of eigenvalues)

An $n \times n$ matrix has at most n-many distinct eigenvalues.

For the maximum number of distinct eigenvalues, the roots actually determine the polynomial!

Fact B (Char. poly. for n-many distinct eigenvalues)

If an $n \times n$ matrix has n-many distinct eigenvalues, then the eigenvalues determine the characteristic polynomial:

$$p_A(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

Example

If we know the eigenvalues of

$$A := \begin{bmatrix} 0 & 3 & -1 \\ -1 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

are $1, 2, 3$, then we can conclude that ...

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$$p_A(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$$

What if there are fewer eigenvalues?

We can try to 'count' eigenvalues with multiplicity, but there are several ways to define this and they do not agree.

We can say a bit more about the coefficients of the char. poly.

The trace of a square matrix

The **trace** of A, denoted tr(A), is the sum of the diagonal entries.

Example

$$\operatorname{tr}\begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 0 + 3 + 2 = 5$$

We won't use the trace often, but it's very easy to compute and it has several nice properties:

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \quad \operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$$

Fact C (Two notable coefficients of the characteristic polynomial)

Let A be an $n \times n$ matrix. Then

- The coefficient of x^n is 1.
- The coefficient of x^{n-1} is $-\operatorname{tr}(A)$.
- The constant term of $p_A(x)$ is $(-1)^n \det(A)$.

Example

$tr(C) = 3+2$
$= 5$

$$C = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$det(C) = 2 \cdot C_{33}$
$= 2 \cdot (-1)^{3+3} \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix}$
$= 2(2)$
$= 4$

$$p_C(x) = 1x^3 - 5x^2 + 8x - 4$$

Note that we don't have a nifty trick to describe the 8.

Recall the factorization when A has n-many distinct eigenvalues.

$$p_A(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

If we multiply out and label the trace and determinant...

$$p_A(x) = x^n - \underbrace{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)}_{\mathrm{tr}(A)} x^{n-1} + \cdots + (-1)^n \underbrace{\lambda_1\lambda_2\cdots\lambda_n}_{\det(A)}$$

...we notice a deep fact!

Fact D (Determinant and trace for n-many distinct eigenvalues)

Let A be an $n \times n$ matrix with n-many distinct eigenvalues.

- The determinant of A is the product of the eigenvalues of A.
- The trace of A is the sum of the eigenvalues of A.

This can be extended to all square matrices by counting eigenvalues with multiplicity, but we won't talk about this (yet).

Exercise 5

Find the characteristic polynomial of

$$A := \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

without computing $\det(x\mathrm{Id} - \mathrm{A})$ directly.

Exercise 6

If we already know that

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 5 & -1 & 1 \end{bmatrix}$$

has three distinct eigenvalues and two of them are -4 and 3, find the last eigenvalue.

Exercise 5

Find the characteristic polynomial of

$$A := \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

without computing $\det(x\mathrm{Id} - A)$ directly.

Solution

Because A is a 2x2 matrix, the characteristic polynomial $P_A(x)$ is a quadratic polynomial

$$P_A(x) = \boxed{a_2} x^2 + \boxed{a_1} x + \boxed{a_0}$$

Fact C: a. The coefficient of x^n is 1

Here $n=2$, so $a_2 = 1$

b. The coefficient of x^{n-1} is $-tr(A)$

so $a_1 = -tr(A)$

$$= -tr\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$= -(3+4)$$

$$= -7$$

c. The constant term of $P_A(x)$ is $(-1)^n \det(A)$

which is $(-1)^2 \det\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = 1 \cdot (3 \cdot 4 - 1 \cdot 2) = 10$

$$\boxed{P_A(x) = 1x^2 - 7x + 10}$$

$$= (x-2)(x-5)$$

Exercise 6

If we already know that

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 5 & -1 & 1 \end{bmatrix}$$

has three distinct eigenvalues and two of them are -4 and 3, find the last eigenvalue.

Solution

- Since $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 5 & -1 & 1 \end{bmatrix}$ has three distinct eigenvalues, Fact D says $\det\left(\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 5 & -1 & 1 \end{bmatrix}\right)$ equals the product of the tree eigenvalues.

- Since two of the eigenvalues are given, the third eigenvalue must be $\dfrac{\det\left(\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 5 & -1 & 1 \end{bmatrix}\right)}{(-4).(3)}$

- $$\det\left(\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 5 & -1 & 1 \end{bmatrix}\right) = 3 \cdot C_{22}$$
$$= 3.(-1)^{2+2} \begin{vmatrix} 1 & 5 \\ 5 & 1 \end{vmatrix}$$
$$= 3 \ (1.1 - 5.5)$$
$$= 3 \ (-24)$$
$$= -72$$

- the third eigenvalue must be $\dfrac{\det\left(\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 5 & -1 & 1 \end{bmatrix}\right)}{(-4).(3)} = \dfrac{-72}{(-4)(3)} = \boxed{6}$