Lecture 8a

# Characteristic Polynomials

# Review: If $\lambda$ is a number, find $\lambda$-eigenvectors if they exist.

**Recall: Eigenvectors and eigenvalues of a matrix**

An **eigenvector** of an $n \times n$ matrix A is a non-zero vector v with

$$\mathrm{A}v = \lambda v$$

for some number $\lambda$, called the eigenvalue of the eigenvector $v$.

By definition, the zero vector is not an eigenvector.

**Recall: Finding eigenvectors with a given eigenvalue**

The $\lambda$-eigenvectors of A are the non-zero solutions to the matrix equation

$$(\mathrm{A} - \lambda\,\mathrm{Id})\mathrm{v} = \vec{0}$$

# Review last lecture: Given a number $\lambda$, try to find $\lambda$-eigenvectors.

## Exercise 1

Find all 2-eigenvectors of

$$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## Exercise 2

Find all 2-eigenvectors of

$$\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

**Exercise 1**

Find all 2-eigenvectors of

$$A := \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Set $\qquad (A - \lambda \, Id)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for $\lambda = 2$

$$\begin{bmatrix} 0-2 & 2 & -1 \\ -1 & 3-2 & 0 \\ 0 & 0 & 2-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \mapsto -2R_2 \quad \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \mapsto R_1 + R_2 \quad \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \mapsto -R_2 \quad \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

no leading 1: Let $y := t$

$$\left.\begin{array}{ll} \text{1st row:} & -2x + 2y - z = 0 \\ \text{2nd row:} & z = 0 \end{array}\right\} \Rightarrow -2x + 2t - 0 = 0 \Rightarrow x = t$$

So $\quad x = t$

$\qquad y = t$

$\qquad z = 0$

Answer: All 2-eigenvectors of $\begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are of the form $\begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$, where $t$ is a nonzero number

**Exercise 2**

Find all 2-eigenvectors of
$$A := \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

Set $(A - \lambda \, \mathrm{Id}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\lambda = 2$

$$\left[\begin{array}{cc|c} 3-2 & 1 & 0 \\ -1 & 3-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 \mapsto R_1 + R_2 \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right]$$

$$\left.\begin{array}{r} x + y = 0 \\ 2y = 0 \end{array}\right\} \qquad \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

<u>Answer</u> The only solution to $(A - 2\,\mathrm{Id}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

which is a zero vector,

so $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$ has <u>no</u> 2-eigenvectors.

## Eigenvalues are rare and special

A matrix only has eigenvectors for a few eigenvalues (or none).

## Eigenvalues of a matrix

The **eigenvalues** of a square matrix A are the numbers $\lambda$ for which there exists a $\lambda$-eigenvector.

Given a matrix A, how can we find the eigenvalues of A? That is, given A and $\lambda$, how can we (easily) tell if $\lambda$-eigenvectors exist?

For example, Ex 2 tells us that $\lambda=2$ is <u>not</u> an eigenvalue of the matrix in Ex 2

Ex 1 tells us that $\lambda=2$ <u>is</u> an eigenvalue of the matrix in Ex 1

## Observation 1

$$(A - \lambda \mathrm{Id})v = \vec{0} \tag{1}$$

always has at least one solution, the zero vector (which doesn't count because eigenvectors are non-zero by definition).

So, A has a $\lambda$-eigenvector if equation (1) has more than one solution.
So, $\lambda$ is an eigenvalue of A if equation (1) has more than one solution.

## Observation 2

Since A is square, so is $(A - \lambda \mathrm{Id})$. Therefore,

$$(A - \lambda \mathrm{Id})v = \vec{0}$$

has a unique solution precisely when $(A - \lambda \mathrm{Id})$ is invertible.
Recall: A square matrix is non-invertible iff its determinant is zero.

So, $\lambda$ **is an eigenvalue of** A **if and only if** $\det(A - \lambda \mathrm{Id}) = 0$.

Useful fact!

## Exercise 3

Find the eigenvalues of the matrix

$$A := \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

by solving the equation $\det(A - x\mathrm{Id}) = 0$.

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Find the eigenvalues of the matrix
$$A := \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$
by solving the equation $\det(A - x\mathrm{Id}) = 0$.

$\lambda$ is an eigenvalue of $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ if and only if $\det\begin{pmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{pmatrix} = 0$.

Set $0 = \det\begin{bmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{bmatrix}$

$$\begin{aligned} 0 &= (1-\lambda)(1-\lambda) - (5)(5) \\ &= 1 - 2\lambda + \lambda^2 - 25 \\ &= \lambda^2 - 2\lambda - 24 \end{aligned}$$

"Complete the square"

$$0 = \lambda^2 - 2\lambda - 24$$

$$+1^2 = \underbrace{\lambda^2 - 2(1)\lambda + 1^2}\ - 24$$

$$1 = (\lambda - 1)^2 \quad - 24$$

$$25 = (\lambda - 1)^2$$

$$\lambda - 1 = 5, -5$$

$$\lambda = 6, -4$$

<u>Answer</u> The eigenvalues of $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ are 6 and $-4$.

Our main tool in the previous exercise is an important idea.

**Definition: The characteristic polynomial of a square matrix**

Given a $n \times n$ matrix A, the function of $x$

$$p_{\mathrm{A}}(x) := \det(x\ \mathrm{Id} - \mathrm{A})$$

is a polynomial of degree $n$, the characteristic polynomial of A.

**Example of a characteristic polynomial**

If $C := \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, the characteristic polynomial of $C$ is

$$P_C(x) = \begin{vmatrix} x-0 & 2 & -1 \\ -1 & x-3 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = (x-2) \cdot C_{33} = (x-2) \cdot (-1)^{3+3} \begin{vmatrix} x & 2 \\ -1 & x-3 \end{vmatrix}$$

$$= (x-2) \cdot 1 \cdot (x \cdot (x-3) + 2)$$

$$= (x-2)(x^2-3x+2)$$

$$= (x-2)(x-1)(x-2)$$

$$= (x-1)(x-2)^2$$

$$= x^3 - 5x^2 + 8x - 4$$

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Given a $n \times n$ matrix A, the function of $x$

$$p_{\mathrm{A}}(x) := \det(x \,\mathrm{Id} - \mathrm{A})$$

is a polynomial of degree $n$, the characteristic polynomial of A.

**Example of a characteristic polynomial**

If $C := \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, the characteristic polynomial of $C$ is

$$\begin{aligned} p_{\mathrm{C}}(x) = \begin{vmatrix} x & -2 & 1 \\ 1 & x-3 & 0 \\ 0 & 0 & x-2 \end{vmatrix} &= (x-2)(-1)^{3+3} \begin{vmatrix} x & -2 \\ 1 & x-3 \end{vmatrix} \\ &= (x-2)(x \cdot (x-3) + 2) \\ &= (x-2)(x^2 - 3x + 2) \\ &= (x-1)(x-2)^2 \\ &= x^3 - 5x^2 + 8x - 4 \end{aligned}$$

Restate our criterion for eigenvalues ($\lambda$ is an eigenvalue of $A$ iff $\det(A - \lambda Id) = 0$) using the characteristic polynomial.

### Finding eigenvalues of A

The eigenvalues of A are the roots of the char. poly. $p_A(x)$ of A.

### Exercise 4(a)

Find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

## Exercise 4(a) solution

The characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ is

$$
\begin{aligned}
c_A(x) &= \det(x\,\mathrm{Id} - A) \\
&= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} x-4 & 2 \\ 1 & x-3 \end{bmatrix}\right) \\
&= (x-4)(x-3) - 2 \\
&= x^2 - 7x + 10 \\
&= (x-2)(x-5)
\end{aligned}
$$

The roots of $c_A(x)$ are 2 and 5.

$\boxed{\text{So } A \text{ has eigenvalues 2 and 5}}$

## Exercise 4(a) solution

The characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ is

$$\begin{aligned} c_A(x) &= \det(x\, Id - A) \\ &= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} x-4 & 2 \\ 1 & x-3 \end{bmatrix} \\ &= (x-4)(x-3)-2 \\ &= x^2-7x+10 \\ &= (x-2)(x-5). \end{aligned}$$

So $A$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.

## Recall how to find eigenvectors

The $\lambda$-eigenvectors $v$ of $A$ are the nonzero solutions to the matrix equation $(\lambda I - A)v = \vec{0}$.

## Exercise 4(b)

Find all eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

## Exercise 4(b) solution

To find the 2-eigenvectors of $A$, solve $(2Id - A)v = \vec{0}$:

$$\left[\begin{array}{rr|r} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x = y$

The eigenvectors with eigenvalue 2 are

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \text{ is a nonzero number .}$$

## Exercise 4(b) solution

To find the 2-eigenvectors of $A$, solve $(2Id - A)v = \vec{0}$:

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x = y$$

The eigenvectors with eigenvalue 2 are

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \text{ is a nonzero number .}$$

To find the 5-eigenvectors of $A$, solve $(5Id - A)v = \vec{0}$:

$$\begin{bmatrix} 5-4 & 2 \\ 1 & 5-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x = -2y$$

The 5-eigenvectors of $A$ are

$$\begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ where } s \text{ is a nonzero number.}$$