Lecture 7b

Eigenvectors

Finding a fixed vector

Given a matrix A, can you find $\vec{x}$ such that

$$A\vec{x} = \vec{x}?$$

That is, can you find a vector $\vec{x}$ which is sent to itself when multiplied by A? Such a vector is called a fixed vector of A.

Exercise 6

$$A := \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

Find all the vectors $\vec{x}$ such that $A\vec{x} = \vec{x}$.

Exercise 6

$$A := \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

Find all the vectors $\vec{x}$ such that $A\vec{x} = \vec{x}$.

Solution

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\left.\begin{aligned} 2x &= x \\ x + 2y - z &= y \\ x + 3y - 2z &= z \end{aligned}\right\}$$

$$\left.\begin{aligned} x &= 0 \\ x + y - z &= 0 \\ x + 3y - 3z &= 0 \end{aligned}\right\}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 3 & -3 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \mapsto -R_1 + R_2$

$R_3 \mapsto -R_1 + R_3$

$R_3 \mapsto -3R_2 + R_3$

Set $z := t$

$y - z = 0$

$y = t$

$x = 0$

Ex 6

Answer: The vectors $\vec{x}$ which satisfy $A\vec{x} = \vec{x}$ can be described by $\begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$, for any t.

Sanity check:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ t \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 2t - t \\ 3t - 2t \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ t \end{pmatrix} \checkmark$$

We can generalize this by including a scaling factor.

Exercise 7

$$A := \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

a. Find all the vectors $\vec{x}$ such that $A\vec{x} = 2\vec{x}$.

b. Find all the vectors $\vec{x}$ such that $A\vec{x} = 3\vec{x}$.

7 a

Exercise 7

$$A := \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

a. Find all the vectors $\vec{x}$ such that $A\vec{x} = 2\vec{x}$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{2\,\mathrm{Id}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

since matrix multiplication distributes over addition

$$\left[\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & -4 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 3 & -4 & 0 \end{array}\right]$$

$\downarrow R_2 \mapsto -R_1 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$\downarrow R_2 \mapsto \frac{1}{3} R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Let $z := t$

$y - z = 0 \implies y = t$

$x - z = 0 \implies x = t$

Ex 7a

Answer: The vectors $\vec{x}$ which satisfy $A\vec{x} = 2\vec{x}$ can be described by $\begin{pmatrix} t \\ t \\ t \end{pmatrix}$ for any t.

For example, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a 2-eigenvector of A.

$\begin{pmatrix} \pi \\ \pi \\ \pi \end{pmatrix}$ is also a 2-eigenvector of A.

The matrix A has infinitely many 2-eigenvectors.

Check:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} t \\ t \\ t \end{pmatrix} = \begin{pmatrix} 2t \\ t + 2t - t \\ t + 3t - 2t \end{pmatrix} = \begin{pmatrix} 2t \\ 2t \\ 2t \end{pmatrix} = 2\begin{pmatrix} t \\ t \\ t \end{pmatrix} \checkmark$$

7b

Exercise 7

$$A := \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

b. Find all the vectors $\vec{x}$ such that $A\vec{x} = 3\vec{x}$.

$$\underbrace{(A - 3\,Id)}_{3\times 3} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{3\times 1} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{3\times 1}$$

$$\left[\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & 3 & -5 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 3 & -5 & 0 \end{array}\right]$$

$R_2 \mapsto R_1 + R_2$
$R_3 \mapsto R_1 + R_3$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right]$$

$R_3 \mapsto 3R_2 + R_3$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right]$$

3rd Row: $-8z = 0 \Rightarrow z = 0$

2nd Row: $-y - z = 0 \Rightarrow y = 0$

1st Row: $-x = 0 \Rightarrow x = 0$

Ex 7b Answer:

The only vector $\vec{x}$ which satisfies $A\vec{x} = 3\vec{x}$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(So A has no 3-eigenvectors)

These questions lead to one of the fundamental ideas of this class.

Definition: Eigenvectors and eigenvalues

An **eigenvector** of a matrix A is a $\boxed{\text{non-zero}}$ vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}$$

for some number λ. The number λ is called the eigenvalue of the eigenvector $\vec{v}$. (lambda)

We also refer to 'an eigenvector with eigenvalue λ' as a **λ-eigenvector**.

Example (from Exercise 7)

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is one of the many 2-eigenvectors of $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$,

but $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$ has no 3-eigenvector.

- The zero vector is not an eigenvector of any matrix by definition.
- An eigenvalue is a number which may be 0 or nonzero.
- If a matrix is not square, can it have eigenvectors?

Finding eigenvectors with a given eigenvalue

The eigenvectors of A with eigenvalue λ are the non-zero solutions to the homogeneous equation

$$(A - \lambda \mathrm{Id})\vec{x} = \vec{0}$$

Exercise 8

$$A := \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

a) Find all the eigenvectors of A with eigenvalue 0.
b) Find one eigenvector of A with eigenvalue 5.

8a

Exercise 8

$$A := \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

a) Find all the eigenvectors of A with eigenvalue 0.

$$\begin{bmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 0$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right]$$ Initial augmented matrix

$R_1 \mapsto \frac{1}{2} R_1$ $$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right]$$

$R_2 \mapsto R_1 + R_2$ $$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y := t$

$2x - y = 0 \Rightarrow 2x = t \Rightarrow x = \frac{1}{2} t$

Answer The 0-eigenvectors of A are of the form

$$\begin{pmatrix} \frac{1}{2} t \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

Check $$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1/2\, t \\ t \end{bmatrix} = \begin{bmatrix} 2t - 2t \\ -t + t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1/2\, t \\ t \end{bmatrix} \checkmark$$

8b

Exercise 8

$$A := \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

a) Find all the eigenvectors of A with eigenvalue 0.
b) Find one eigenvector of A with eigenvalue 5.

$$(A - \lambda \, Id)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 4-\lambda & -2 & 0 \\ -2 & 1-\lambda & 0 \end{array}\right]$$

$\lambda = 5$
$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right]$$

$R_1 \mapsto -R_1$, $R_2 \mapsto \frac{1}{2} R_2$
$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & 0 \end{array}\right]$$

$R_2 \mapsto R_1 + R_2$
$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x + 2y = 0$

Let $y := t$

$x + 2t = 0 \implies x = -2t$

The 5-eigenvectors of A are of the form

$\begin{pmatrix} -2t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ for any t.

Answer $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ works.

Check
$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \cdot -2 + -2 \cdot 1 \\ -2 \cdot -2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \checkmark$$

Eigenvalues are rare and special

A matrix will only have eigenvectors for a few special eigenvalues.

Next time

How to find the λs for which a λ-eigenvector exist.

(I hope you like finding roots of polynomials.)