## Lecture 7a

# Cofactors

## Recall: The key properties of the determinant

1. $A$ is invertible if and only if $\det(A) \neq 0$.
2. If $A$ and $B$ are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B)$$

## Recall: How to compute the determinant

- Approach 1: Use row operations to relate it to a determinant we know.
  - Row operations change determinant in simple ways.
  - The determinant of an upper triangular matrix is the product of diagonal entries.
- Use a formula for the determinants of small matrices.
  - $2 \times 2$ matrices: the determinant is $ad - bc$.
  - $3 \times 3$ matrices: Sarrus' rule.

# Cofactors

We will learn one more method to calculate determinants.

## Definition: Cofactors of a square matrix

The **$(i,j)$th cofactor** of A is

$$c_{i,j} := (-1)^{i+j}\left(\begin{array}{c}\text{the determinant of the matrix obtained by}\\ \text{deleting the } i\text{th row and } j\text{th column of A}\end{array}\right)$$

## Example: The $(2,1)$-cofactor of a $3 \times 3$ matrix

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then $c_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = (-1)(18-24) = \mathbf{6}$

1st col

2nd row $\begin{array}{ccc} \cancel{1} & 2 & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ \cancel{7} & 8 & 9 \end{array}$

## How to remember the sign of the cofactor

The sign $(-1)^{i+j}$ in the cofactor is positive in the upper left entry, and alternates in a checkerboard pattern.

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

## Exercise 1

Compute the $(3,2)$-cofactor of the following matrix.

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 1 \\ 1 & 1 & 2 & 5 \end{bmatrix}$$

Exercise 1 (solution):
$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{matrix} 2 & 3 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 1 \\ 1 & 1 & 2 & 5 \end{matrix}$$

3rd row

2nd col

$$c_{3,2} = (-1)^{3+2} \begin{vmatrix} 2 & 1 & 1 \\ 0 & -1 & 3 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= (-1) \cdot - \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 0 & -3 & -9 \end{vmatrix} \qquad \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \mapsto \begin{bmatrix} R_1 \\ R_2 \\ -2R_1 + R_3 \end{bmatrix}$$

does not change the determinant

$$= -3 \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 0 & 1 & 3 \end{vmatrix} \qquad \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \mapsto \begin{bmatrix} R_1 \\ R_2 \\ -\frac{1}{3} R_3 \end{bmatrix}$$

$$= -3 \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 0 & 0 & 6 \end{vmatrix} \qquad \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \mapsto \begin{bmatrix} R_1 \\ R_2 \\ R_2 + R_3 \end{bmatrix}$$

does not change the determinant

Exercise 1 (solution):
$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{matrix} 2 & 3 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 1 \\ 1 & 1 & 2 & 5 \end{matrix}$$

3rd row

2nd col

$$c_{3,2} = (-1)^{3+2} \begin{vmatrix} 2 & 1 & 1 \\ 0 & -1 & 3 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= (-1) \cdot - \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 0 & -3 & -9 \end{vmatrix} \qquad \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \mapsto \begin{bmatrix} R_1 \\ R_2 \\ -2R_1 + R_3 \end{bmatrix}$$

does not change the determinant

$$= -3 \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 0 & 1 & 3 \end{vmatrix} \qquad \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \mapsto \begin{bmatrix} R_1 \\ R_2 \\ -\frac{1}{3} R_3 \end{bmatrix}$$

$$= -3 \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 0 & 0 & 6 \end{vmatrix} = -3(1 \cdot -1 \cdot 6) = \boxed{18}$$

# Computing det($A$) using cofactor expansion

**Computing det(A), approach 2: Cofactor expansion**

If A is an $n \times n$ matrix, we can compute its determinant as follows. The cofactor expansion of det(A) along the $i$th row is

$$\det(\mathrm{A}) = \sum_{j=1}^{n} a_{ij} c_{ij} = a_{i1}c_{i1} + a_{i2}c_{i2} + a_{i3}c_{i3} + a_{i4}c_{i4} + \cdots + a_{in}c_{in}$$

Here, $a_{ij}$ denotes the $(i, j)$th entry of A.

row $i$ $\quad [a_{i,1} \quad a_{i,2} \quad \cdots \quad a_{i,n}]$

## Example

Cofactor expansion along the 1st row:

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \end{array} = \overset{+}{1} \overbrace{\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}}^{C_{11}} + 2 \cdot \overbrace{\overset{-}{-} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}}^{C_{12}} + \overset{+}{3} \overbrace{\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}}^{C_{13}}$$

$$= 1 \cdot (-3) + 2 \cdot -(-6) + 3 \cdot (-3) = 0$$

## Example

Cofactor expansion along the 2nd row:

### Example

Cofactor expansion along the 1st row:

$$\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2 \cdot - \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 \cdot (-3) + 2 \cdot -(-6) + 3 \cdot (-3) = 0$$

### Example

Cofactor expansion along the 2nd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ 7 & 8 & 9 \end{vmatrix} = 4 \overbrace{\cdot - \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}}^{C_{21}} + 5 \overbrace{\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}}^{C_{22}} + 6 \overbrace{\cdot - \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}}^{C_{23}}$$

$$= 4 \cdot -(-6) + 5 \cdot (-12) + 6 \cdot -(-6) = 0$$

Intuitively, we travel along the *i*th row and take the alternating sum of each entry times the determinant of the complement.

Since $\det(A^\top) = \det(A)$, we can use columns instead of rows.

## Cofactor expansion along a column

The cofactor expansion of $\det(A)$ along the $j$th column is

$$\det(A) = \sum_{i=1}^{n} a_{ij}c_{ij} = \overset{\text{row 1, col } j}{a_{1j}}c_{1j} + \overset{\text{row 2, col } j}{a_{2j}}c_{2j} + a_{3j}c_{3j} + \cdots + a_{nj}c_{nj}$$

## Example

Cofactor expansion along the 2nd column:

$$\begin{vmatrix} 1 & \mathbf{2} & 3 \\ 4 & \mathbf{5} & 6 \\ 7 & \mathbf{8} & 9 \end{vmatrix} = \quad \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} \quad \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} \quad \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}$$

$$= \quad \cdot (\quad) \quad \cdot (\quad) \quad \cdot (\quad) =$$

Since $\det(A^\top) = \det(A)$, we can use columns instead of rows.

## Cofactor expansion along a column

The cofactor expansion of $\det(A)$ along the $j$th column is

$$\det(A) = \sum_{i=1}^{n} a_{ij}c_{ij} = \overset{\text{row 1, col } j}{a_{1j}}c_{1j} + \overset{\text{row 2, col } j}{a_{2j}}c_{2j} + a_{3j}c_{3j} + \cdots + a_{nj}c_{nj}$$

## Example

Cofactor expansion along the 2nd column:

$$\begin{vmatrix} ^{+}1 & \bar{\mathbf{2}} & 3 \\ 4 & ^{+}\mathbf{5} & 6 \\ 7 & ^{-}\mathbf{8} & 9 \end{vmatrix} = \overbrace{2(-1)\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}}^{C_{12}} + \overbrace{5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}}^{C_{22}} + \overbrace{8(-1)\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}^{C_{32}}$$

$$= 2^{(-1)}(-6) + 5 \cdot (-12) + 8^{(-1)}(-6) = 0$$

## Exercise 2

Find $\det A$ for $A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}$.

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**Solution.**

Using cofactor expansion along column 3, $\det A = 0$.

## Fact

*If $A$ is an $n \times n$ matrix with a row or column of zeros, then $\det A = 0$.*

## Exercise 3

Compute the determinant of

$$\begin{bmatrix} 1 & -1 & 7 & 0 & 1 \\ 0 & 2 & 6 & 0 & 1 \\ 7 & 5 & -6 & 2 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix}$$

## Exercise 3

Compute the determinant of

$$A := \begin{bmatrix} 1^{+} & -1^{-} & 7^{+} & 0^{-} & 1 \\ 0 & 2 & 6 & 0^{+} & 1 \\ 7 & 5 & -6 & 2^{-} & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix}$$

<u>Answer</u>

$$\det A = a_{3,4}\, C_{3,4}$$

$$= 2\,(-1) \begin{vmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 4 & 3 \end{vmatrix}$$

## Exercise 3

Compute the determinant of

$$A := \begin{bmatrix} 1 & -1 & 7 & 0 & 1 \\ 0 & 2 & 6 & 0 & 1 \\ 7 & 5 & -6 & 2 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix}$$

Answer

$$\det A = a_{3,4}\, C_{3,4}$$

$$= 2 \cdot -1 \begin{vmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 4 & 3 \end{vmatrix}$$

$$= -2 \cdot (-1)(+1) \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{vmatrix}$$

## Exercise 3

Compute the determinant of

$$A := \begin{bmatrix} 1^{+} & -1^{-} & 7^{+} & 0^{-} & 1 \\ 0 & 2 & 6 & 0^{+} & 1 \\ 7 & 5 & -6 & 2^{-} & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \end{bmatrix}$$

**Answer**

$$\det A = a_{3,4}\, C_{3,4}$$

$$= 2 \cdot -1 \begin{vmatrix} 1 & -1 & 7 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 4 & 3 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 2 \cdot \left[ 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + 0 + 1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \right]$$

$$= 2\left[(6 - -1) + (-1 \cdot 1 - 2 \cdot 1)\right]$$

$$= 2\left[6 + 1 - 1 - 2\right]$$

$$= \boxed{8}$$

## When to use cofactor expansion?

Cofactor expansion takes advantage of 0s in a fixed row or column.

## Exercise 4

Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

## Weaknesses of cofactor expansion

Without clever tricks, this is much slower than using row operations.

## Exercise 4 (solution)

Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

Cofactor expansion along row 1 yields

$$\begin{aligned} \det(A) &= 0 \cdot c_{11}(A) + 1 \cdot c_{12}(A) + 2 \cdot c_{13}(A) + 1 \cdot c_{14}(A) \\ &= 1c_{12}(A) + 2c_{13}(A) + c_{14}(A), \end{aligned}$$

whereas cofactor expansion along, row 3 yields

$$\begin{aligned} \det(A) &= 0 \cdot c_{31}(A) + 1 \cdot c_{32}(A) + (-1) \cdot c_{33}(A) + 0 \cdot c_{34}(A) \\ &= 1 \cdot c_{32}(A) + (-1) \cdot c_{33}(A), \end{aligned}$$

i.e., in the first case we have to compute three cofactors, but in the second we only have to compute two.

## Exercise 4 (solution) con't

Save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1c_{32}(A) + (-1)c_{33}(A) \\ &= 1(-1)^5 \begin{vmatrix} 0 & 2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1)(-1)^{?} \begin{vmatrix} & & \\ & & \\ & & \end{vmatrix} \end{aligned}$$

## Exercise 4 (solution) con't

Save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1c_{32}(A) + (-1)c_{33}(A) \\ &= 1(-1)^5 \begin{vmatrix} 0 & 2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1)(-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \end{aligned}$$

## Exercise 4 (solution) con't

Save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned}
\det A &= 1c_{32}(A) + (-1)c_{33}(A) \\
&= \underbrace{1(-1)^5}\begin{vmatrix} 0 & 2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + \underbrace{(-1)(-1)^6}\begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= \underbrace{(-1)\,2\,(-1)^3}\underbrace{\begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix}} + \underbrace{(-1)\,1\,(-1)^3}\underbrace{\begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix}} \\
&= 2\,(5.2-7.3) + 1 \cdot (11) \\
&= 2 \cdot (-11) + (-11) \\
&= \boxed{-33}
\end{aligned}$$

## Exercise 4 (solution) con't

Save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned}
\det A &= 1c_{32}(A) + (-1)c_{33}(A) \\
&= 1(-1)^5 \begin{vmatrix} 0 & 2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1)(-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= (-1)2(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} + (-1)1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= 2(10-21) + 1(10-21) \\
&= 2(-11) + (-11) \\
&= \boxed{-33}.
\end{aligned}$$

## Exercise 4 con't

Try computing $\det \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$ using cofactor expansion along other convenient rows and columns (column 1, 2, or 3; row 2 or 4).

You should still get $\det A = -33$.

Remarkably, cofactors can also be used to compute matrix inverses!

**Classical formula: Compute the inverse of a matrix using cofactors**

Let A be an $n \times n$ matrix with non-zero determinant. Then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ c_{1,2} & c_{2,2} & \cdots & c_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1,n} & c_{2,n} & \cdots & c_{n,n} \end{bmatrix}$$

**The subscripts are not a mistake!**

The $(i, j)$th entry of the inverse is the $(j, i)$-cofactor of A, divided by the determinant.

Note: If you write a code for computing the inverse of an invertible matrix, you would not use this formula because it's very inefficient (you would need to compute $\det(A)$ and also $n^2$ more $n-1 \times n-1$ determinants).

## Exercise 5

(a) Find the first column of $A^{-1}$, where

$$A := \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

(b) Check your work by computing $AA^{-1}$.

## Exercise 5

**Solution**

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Compute $\det(A)$ by taking the cofactor expansion along the 1st row:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$C_{11} = +\begin{vmatrix} 4 & -1 \\ 3 & 0 \end{vmatrix} = 0 - -3 = 3$$

$$C_{12} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0 - -1) = -1$$

$$C_{13} = +\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3 - 4 = -1$$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2 \cdot 3 + 7 \cdot -1 + 1 \cdot -1 \\ &= 6 - 7 - 1 \\ &= -2 \end{aligned}$$

a) The 1st column of $A^{-1}$ is

$$\frac{1}{\det(A)} \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

b) Check: $AA^{-1}$ should be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $A\,[\text{1st col of } A^{-1}]$ should be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} -3/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \cdot \frac{-3}{2} + 7 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \\ 1 \cdot -3/2 + 4 \cdot \frac{1}{2} + -1 \cdot \frac{1}{2} \\ 1 \cdot -\frac{3}{2} + 3 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

## Recall: The key properties of the determinant

1. $A$ is invertible if and only if $\det(A) \neq 0$.
2. If $A$ and $B$ are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B)$$

## Summary: How to compute the determinant

- Approach 1: Use row operations to relate it to an upper triangular matrix.
- Use a formula for the determinants of small matrices.
  - $2 \times 2$ matrices: the determinant is $ad - bc$.
  - $3 \times 3$ matrices: Sarrus' rule.
- Approach 2 (new): use cofactor expansion