Lecture 6b

Determinants, second part

Last time: The determinant of a square matrix

For each square matrix A, we have a number $\det(A)$ which satisfies:

i. A is invertible if and only if $\det(A) \neq 0$.
ii. $\det(AB) = \det(A)\det(B)$
iii. $\det(Id) = 1$

Compute $\det(A)$ by first turning A into an upper triangular matrix and keeping track of how the determinants change.

Goal

- Additional properties of det
- Sarrus' Rule, a method for computing the determinant of a 3×3 matrix.

Exercise 8

Show that there is a unique solution to the following system.

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 12 \\
3x_2 &= x_1 + x_3 \\
x_1 + 2x_3 &= 6 + 2x_2
\end{aligned}
$$

Use the following strategy: Write it as a system of linear equations and compute the determinant of the coefficient matrix.

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(Answer to Exercise 8) The system can be rewritten as follows.

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 12 \\
x_1 - 3x_2 + x_3 &= 0 \\
x_1 - 2x_2 + 2x_3 &= 6
\end{aligned}
$$

The coefficient matrix is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & -2 & 2 \end{bmatrix}$.

(Answer to Exercise 8 con't) First, we compute the determinant of the coefficient matrix.

$R_2 \mapsto -R_1 + R_2$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -4 & 0 \\ 1 & -2 & 2 \end{vmatrix}$$

$R_3 \mapsto -R_1 + R_3$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -4 & 0 \\ 0 & -3 & 1 \end{vmatrix}$$

$R_2 \mapsto -\frac{1}{4} R_2$

$$= -4 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{vmatrix}$$

Multiplying a row by c multiplies the determinant by c

$R_3 \mapsto 3R_2 + R_3$

$$= -4 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -4 \cdot 1 = \boxed{-4}.$$

(Answer to Exercise 8 con't)

Recall Property i of det: M is invertible if and only if $\det(M) \neq 0$.

- Let $C := \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & -2 & 2 \end{bmatrix}$. Since we $\det(C) = -4 \neq 0$, Property i tells us that the inverse C^{-1} exists.
- The linear system is equivalent to

$$C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 6 \end{bmatrix}.$$

$$\underbrace{C^{-1}C}_{Id} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C^{-1} \begin{bmatrix} 12 \\ 0 \\ 6 \end{bmatrix}$$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C^{-1} \begin{bmatrix} 12 \\ 0 \\ 6 \end{bmatrix}$ gives the unique solution.

We have shown that the system in Exercise 8 has a unique solution.

Remark: The fact that the determinant of the coefficient matrix is non-zero tells us that the linear system has a unique (exactly one) solution. (We don't need to check the constant terms in the system!)

The determinant of the transpose

$$\det(A^\top) = \det(A)$$

Example

$$\begin{vmatrix} 1 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 1 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

This identity is remarkably useful, since it allows us to deduce new determinant identities from old ones.

Anything we can do with rows, we can do with columns

We can use the transpose to deduce that column operations change the determinant in simple ways.

Example

We can swap two columns by swapping rows in the transpose:

$$\begin{vmatrix} 1 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \qquad \begin{vmatrix} 4 & 1 & 7 \\ 5 & 1 & 8 \\ 6 & 3 & 9 \end{vmatrix}$$

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$|A| = |A^T|$ swapping two rows multiplies the determinant by -1 $|A| = |A^T|$

- Swapping two columns multiplies the determinant by -1.

There is a general formula for the determinant of an $n \times n$ matrix.

Computing determinants: The general formula

If A is an $n \times n$ matrix, then

$$\det(A) = \sum (-1)^{s(p)} a_{p_1,1} a_{p_2,2} \cdots a_{p_i,i} \cdots , a_{p_n,n}$$

where the sum runs over all ways to list the numbers $1, 2, ..., n$ in some order as $p_1, p_2, ..., p_n$, and $s(p)$ is the number of pairs i, j such that $i < j$ but $p_i > p_j$.

- This formula lies under the hood of all our previous results.
- It impractical for computation (both for you and for a computer). For a 4×4 matrix, it already has $4! = 24$ terms!

In practice, you (the student) and computers use more efficient methods to calculate determinants.

(Only works for 3×3 matrix, but worth knowing about. Often used in the cross product of two 3-vectors.)

Computing det(A), special case for 3×3 ONLY (Sarrus' rule)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Sarrus' rule trick: Copy the first two columns to the right of the matrix: Add the product of the elements in each diagonal.

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \quad a \cdot e \cdot i + b \cdot f \cdot g + c \cdot d \cdot h$$

Then, subtract the product of the elements in each antidiagonal.

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \quad - g \cdot e \cdot c - h \cdot f \cdot a - i \cdot d \cdot b$$

The result is the determinant of the original matrix.

Example using Sarrus' rule

$$\text{Compute } \det\left(\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}\right).$$

Add the product of the elements in each diagonal.

$$\left|\begin{array}{ccc|cc}1 & 2 & 3 & 1 & 2\\ 4 & 5 & 6 & 4 & 5\\ 7 & 8 & 9 & 7 & 8\end{array}\right. \quad 1\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8 = 225$$

Then, subtract the product of the elements in each antidiagonal.

$$\left|\begin{array}{ccc|cc}1 & 2 & 3 & 1 & 2\\ 4 & 5 & 6 & 4 & 5\\ 7 & 8 & 9 & 7 & 8\end{array}\right. \quad -7\cdot 5\cdot 3 - 8\cdot 6\cdot 2 - 9\cdot 4\cdot 1 =$$

The determinant of the original matrix is $\boxed{\mathbf{0}}$.