Lecture 6a

Determinants

Recall: The determinant of a 2×2 matrix

Given a 2×2 matrix

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

its determinant is the number $\det(A) := ad - bc$.

Recall: The 2×2 determinant detects invertibility

A 2×2 matrix A is invertible if $ad - bc$ is not zero, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Goal

Generalizing determinants to all square matrices.

$\det(A)$ is defined for any square matrix! It's always a number.

Determinants (the idea)

For each square matrix A, we can define a number det(A) called the **determinant of** A which satisfies two fundamental properties:

i. Property i: A is invertible if and only if $\det(A) \neq 0$.
ii. Property ii: $\det(AB) = \det(A)\det(B)$.

Property i implies the following:

- If the inverse of A exists, then det(A) is a non-zero number.
- If det(A) is a non-zero number, then the inverse of A exists.
- If A has no inverse, then $\det(A) = 0$.
- If $\det(A) = 0$, then A has no inverse.

Exercise 1

Check that $\det(AB) = \det(A)\det(B)$ for

$$A := \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad B := \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Exercise 1

Check that $\det(AB) = \det(A)\det(B)$ for

$$A := \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad B := \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\det(AB) = \det(\begin{bmatrix} 1 & 8 \\ 3 & 4 \end{bmatrix}) = 1.4 - 8.3 = -20$$

$$\det(A) = 3.2 - 2.1 = 4$$

$$\det(B) = -1.1 - 2.2 = -5$$

$$\det(A)\det(B) = 4 \cdot -5 = -20.$$

Determinant notation

The determinant of an explicit matrix is often denoted by replacing the brackets by vertical lines.

Example

$$\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} := \det\left(\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\right)$$

LHS := RHS

Aside: Notation for two flavors of equality

- $:=$ defines the left side to be equal to the right side.
- $=$ asserts the two sides are equal.

The latter is a statement of fact, the former is a definition.

We first explore the properties of the determinant, and then use them to compute det(A).

Determinant: Identity matrices

iii. Property iii. The determinant of an identity matrix is 1.

The three properties completely determine the determinant and all its properties.

Exercise 2

Show that, if M is invertible, then $\det(M^{-1}) = \frac{1}{\det(M)}$

$1 = \det(Id)$ since the determinant of an identity matrix is 1 (Property iii)

$= \det(MM^{-1})$ by definition of the inverse matrix

$= \det(M)\det(M^{-1})$ by property ii of det

$\det(M^{-1}) = 1/\det(M)$ makes sense since $\det(M) \neq 0$ due to Property i.

For example, if $\det(M) = 7$ then M^{-1} exists and $\det(M^{-1}) = \frac{1}{7}$.

Determinants also play nicely with row operations.

Determinants and row operations

1. Swapping two rows multiplies the determinant by -1.
2. Multiplying a row by c multiplies the determinant by c.
3. Adding a multiple of one row to another row does not change the determinant.

Examples of elementary row operations

1. Swapping two rows **swaps the sign of the determinant**.

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \mapsto \begin{bmatrix} R_2 \\ R_1 \\ R_3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = - \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}$$

2. Dividing a row by a number c **pulls out a factor of c**.

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \mapsto \begin{bmatrix} R_1 \\ \frac{1}{4}R_2 \\ R_3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 1 & \frac{5}{4} & \frac{3}{2} \\ 7 & 8 & 9 \end{vmatrix}$$

3. Adding a multiple of one row to another row **does nothing**.

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \mapsto \begin{bmatrix} R_1 \\ -4R_1 + R_2 \\ R_3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix}$$

Strategy to compute a determinant

Use row operations to transform an unknown determinant into a known one, keeping track of the changes along the way.

Since we can use row operations to put any matrix into REF...

What is the determinant of a matrix in REF?

We can answer a more general question.

A few special types of square matrices

- An **upper triangular matrix** has 0s below the diagonal.
- An **lower triangular matrix** has 0s above the diagonal.
- A **diagonal matrix** has 0s away from the diagonal.

Examples

Upper triangular:

$$\begin{bmatrix} 1 & 2 & 0 \\ \mathbf{0} & 5 & 2 \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}$$

diagonal →

Lower triangular:

$$\begin{bmatrix} 2 & \mathbf{0} & \mathbf{0} \\ -1 & 0 & \mathbf{0} \\ 3 & 0 & -1 \end{bmatrix}$$

Diagonal:

$$\begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}$$

No restriction for numbers on the diagonal.

Note: a diagonal matrix is both an upper & lower triangular matrix.

Note: By def, a square matrix in REF is an upper triangular matrix which has 1s and 0s on the diagonal.

Not every upper triangular matrix is in REF, though.

Determinant: Upper triangular matrices

The determinant of an upper triangular matrix is the product of the diagonal entries.

The determinant of a lower triangular matrix or a diagonal matrix is also the product of the diagonal entries.

Example

$$\begin{vmatrix} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{2} & 4 & 8 \\ 0 & 0 & \mathbf{3} & 9 \\ 0 & 0 & 0 & \mathbf{5} \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdot 5 = 30$$

$$\begin{vmatrix} \mathbf{1} & 2 & 0 \\ 0 & \mathbf{1} & 8 \\ 0 & 0 & \mathbf{0} \end{vmatrix} = 1 \cdot 1 \cdot 0 = 0$$

Computing det(A) using row operations

Computing det(A), approach 1: Put the matrix into REF

Use Gaussian Elimination to put A into REF B. Keep track of how the determinant changes in a number r, so you know $\det(A) = r \det(B)$.

- If B has a row of zeroes, then $\det(B) = 0$ and so $\det(A) = 0$.
- Otherwise, $\det(B) = 1$ so $\det(A) = r$.

This method is consistent and practical (computers use it), but other methods may be easier for hand-computation.

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- If B has a row of zeroes, then $\det(B) = 0$ and so $\det(A) = 0$.
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This method is consistent and practical (computers use it), but other methods may be easier for hand-computation.

approach 1': Put the matrix into upper triangular form

Row reduce to put A into an upper triangular matrix B. Keep track of how the determinant changes in a number r, so $\det(A) = r\det(B)$.

- If B has a row of zeroes, then $\det(B) = 0$ and so $\det(A) = 0$.
- Otherwise, $\det(A) = r \cdot$ (product of diagonal entries of B).

In practice, you can stop earlier if the determinant is clear.

Exercise 3

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det(A)$ with row operations.

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Approach 1'

swap R_1 & R_2; $R_2 \mapsto -3R_1 + R_2$, $R_3 \mapsto -R_1 + R_3$

$$\det(A) = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & -7 \\ 0 & 3 & 1 \end{vmatrix}$$

$R_2 \mapsto -R_3 + R_2$; $R_3 \mapsto -3R_2 + R_3$

$$= - \begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & -8 \\ 0 & 3 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -5 \\ 0 & 1 & -8 \\ 0 & 0 & 25 \end{vmatrix} = -1 \cdot 1 \cdot 25 = \boxed{-25}$$

Exercise 3

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det(A)$ with row operations.

Approach 1

swap R_1 & R_2

$R_2 \mapsto -3R_1 + R_2$

$R_3 \mapsto -R_1 + R_3$

$$\det(A) = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & -7 \\ 0 & 3 & 1 \end{vmatrix}$$

$R_2 \mapsto -R_3 + R_2$

$R_3 \mapsto -3R_2 + R_3$

$R_3 \mapsto \frac{1}{25} R_3$

$$= -\begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & -8 \\ 0 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -5 \\ 0 & 1 & -8 \\ 0 & 0 & 25 \end{vmatrix} = -25 \begin{vmatrix} 1 & 0 & -5 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -25 \cdot 1 \cdot 1 \cdot 1 = \boxed{-25}$$

Exercise 4

Find $\begin{vmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{vmatrix}$ using row operations (approach 1 or 1').

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$$\begin{vmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$R_2 \mapsto 2R_1 + R_2$

$R_3 \mapsto R_1 + R_3$

Exercise 4

Find $\begin{vmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{vmatrix}$ using row operations (approach 1 or 1').

$$\begin{vmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot 1 \cdot 0 = \boxed{0}.$$

$R_2 \mapsto 2R_1 + R_2$

$R_3 \mapsto R_1 + R_3$

$R_3 \mapsto -R_2 + R_3$

Exercise 5

Find the determinant of $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$ using row operations.

Exercise 5

Find the determinant of $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$ using row operations.

$R_7 \longmapsto R_5 + R_7$

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= -2 \cdot 5 = \boxed{-10}.$$

Exercise 6

Compute $\begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{vmatrix}$ using row operations.

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{vmatrix}$$

Exercise 6

Compute $\begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{vmatrix}$ using row operations.

$R_1 \mapsto R_4$
$R_4 \mapsto R_1$

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{vmatrix}$$

$R_7 \mapsto R_5 + R_7$

$$= - \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{vmatrix}$$

$$= -(5 \cdot 4) = \boxed{-20}.$$

Exercise 7

Compute $\begin{vmatrix} \pi & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 1 & 0 & 4 \\ 3 & 0 & 1 & 7 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 3 & 0 & 1 & 7 & 0 & 6 & 2 \\ 1 & 1 & 1 & 0 & 0 & 2 & 2 \end{vmatrix}$ using row operations.

Exercise 7

Compute $\begin{vmatrix} \pi & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 1 & 0 & 4 \\ 3 & 0 & 1 & 7 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 3 & 0 & 1 & 7 & 0 & 6 & 2 \\ 1 & 1 & 1 & 0 & 0 & 2 & 2 \end{vmatrix}$ using row operations.

$R_6 \mapsto -R_4 + R_6$

$$\begin{vmatrix} \pi & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 1 & 0 & 4 \\ 3 & 0 & 1 & 7 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 3 & 0 & 1 & 7 & 0 & 6 & 2 \\ 1 & 1 & 1 & 0 & 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} \pi & 0 & 0 & 1 & 1 & 5 & 7 \\ 0 & 5 & 0 & 0 & 1 & 1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 1 & 0 & 4 \\ 3 & 0 & 1 & 7 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 & 2 \end{vmatrix}$$

$= \boxed{0}$ because of the row of zeros.

Recap

Determinants

For each square matrix A, we have a number $\det(A)$ which satisfies:

i. A is invertible if and only if $\det(A) \neq 0$.
ii. $\det(AB) = \det(A)\det(B)$
iii. $\det(Id) = 1$

Computing det using row operations

Compute $\det(A)$ by first turning A into an upper triangular matrix while keeping track of how the determinants change.